10.2.3 Coriolis force: $-2m\omega \times \mathbf{v}$

While the centrifugal force is a very intuitive concept (we've all gone around a corner in a car), the same thing cannot be said about the Coriolis force. This force requires a nonzero velocity $\mathbf{v}$ relative to the accelerating frame, and people normally don't move with an appreciable $\mathbf{v}$ with respect to their car while rounding a corner. To get a feel for this force, let's look at two special cases.

Case 1 (Moving radially on a carousel): A carousel rotates counterclockwise with constant angular speed ω. Consider someone walking radially inward on the carousel (imagine a radial line painted on the carousel; the person walks along this line), at speed v with respect to the carousel, at radius r. The angular velocity vector $\boldsymbol{\omega}$ points out of the page, where we've signified the "out" direction in Fig. 10.6 by a little circle with a dot inside.

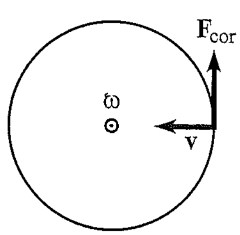

Fig. 10.6

REMARK: This remark might be a little picky, but I'll say it anyway. The direction of rotation is sometimes denoted by a curved arrow pointing tangentially along the circumference of the carousel. But this is technically not correct, because it would imply that the carousel is rotating in the rotating frame, which it isn't; it's just sitting there. And it's understood that Fig. 10.6 is in fact drawn in the rotating frame, and not the lab frame, because it includes a fictitious force, which has nothing to do with the lab frame. (If you wanted to draw things in the lab frame, then you wouldn't draw any fictitious forces, and the velocity **v** would have a tangential component, at least in this setup.) ♣

The Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, points tangentially in the direction of the motion of the carousel, that is, to the person's right in our scenario. It has magnitude

$$F_{\text{cor}} = 2m\omega v. \tag{10.14}$$

The person will have to counter this force with a tangential friction force of $2m\omega v$ (pointing to his left) at his feet, so that he continues to walk on the same radial line. Note that there is also the centrifugal force, which is countered by a radial friction force at the person's feet. But this effect won't be important here.

Why does this Coriolis force exist? It exists so that the resultant friction force changes the angular momentum of the person (measured with respect to the lab frame) in the proper way, according to $\tau = dL/dt$. To see this, take d/dt of $L = mr^2\omega$, where ω is the person's angular speed with respect to the lab frame, which is also the carousel's angular speed. Using $dr/dt = -v$, we have

$$\frac{dL}{dt} = -2mr\omega v + mr^2(d\omega/dt). \tag{10.15}$$

But $d\omega/dt = 0$, because the person remains on one radial line, and we are assuming that the carousel is arranged to keep a constant ω. Equation (10.15) then gives $dL/dt = -2mr\omega v$. So the L (with respect to the lab frame) of the person changes at a rate $-(2m\omega v)r$. This is simply the radius times the tangential friction force applied by the carousel. In other words, it is the torque applied to the person.

REMARK: What if the person doesn't apply a tangential friction force at his feet? Then the Coriolis force of $2m\omega v$ produces a tangential acceleration of $2\omega v$ in the rotating frame, and hence also in the lab frame (initially, before the direction of the motion in the rotating frame has a chance to change), because the frames are related by a constant ω. This acceleration exists essentially to keep the person's angular momentum (with respect to the lab frame) constant. (It *is* constant in this scenario, because there are no tangential forces in the lab frame.) To see that this tangential acceleration is consistent with conservation of angular momentum, set $dL/dt = 0$ in Eq. (10.15) to obtain $2\omega v = r(d\omega/dt)$ (this is the person's ω here, which is changing). The right-hand side of this is by definition the tangential acceleration. Therefore, saying that L is conserved is the same as saying that $2\omega v$ is the tangential acceleration (for this situation where the inward radial speed is v). ♣

Case 2 (Moving tangentially on a carousel): Now consider someone walking tangentially on a carousel in the direction of the carousel's motion, with speed v (relative to the carousel) at constant radius r (see Fig. 10.7). The Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points radially outward with magnitude $2m\omega v$. Assume that the person applies the friction force necessary to continue moving at radius r.

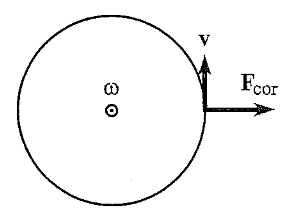

Fig. 10.7

There is a simple way to see why this outward force of $2m\omega v$ exists. Let $V \equiv \omega r$ be the speed of a point on the carousel at radius r, as viewed by an outside observer. If the person moves tangentially (in the same direction as the spinning) with speed v relative to the carousel, then his speed as viewed by the outside observer is $V + v$. The outside observer therefore sees the person walking in a circle of radius r at speed

$V + v$. The acceleration of the person with respect to the ground frame is therefore $(V + v)^2/r$. This acceleration must be caused by an inward-pointing friction force at the person's feet, so

$$F_{\text{friction}} = \frac{m(V+v)^2}{r} = \frac{mV^2}{r} + \frac{2mVv}{r} + \frac{mv^2}{r}. \tag{10.16}$$

This friction force is the same in any frame. How, then, does our person on the carousel interpret the three pieces of the inward-pointing friction force in Eq. (10.16)? The first term balances the outward centrifugal force due to the rotation of the frame, which he always feels. The third term is the inward force his feet must apply if he is to walk in a circle of radius r at speed v, which is exactly what he is doing in the rotating frame. The middle term is the additional inward friction force he must apply to balance the outward Coriolis force of $2m\omega v$ (using $V \equiv \omega r$). Said in an equivalent way, the person on the carousel will write down an $F = ma$ equation of the form (taking radially inward to be positive),

$$\begin{aligned} m\frac{v^2}{r} &= \frac{m(V+v)^2}{r} - \frac{mV^2}{r} - \frac{2mVv}{r} \\ \Longrightarrow \quad m\mathbf{a} &= \mathbf{F}_{\text{friction}} + \mathbf{F}_{\text{cent}} + \mathbf{F}_{\text{cor}}. \end{aligned} \tag{10.17}$$

We see that the net force he feels does indeed equal his ma, where a is measured with respect to the rotating frame. Physically, the difference between the interpretations of Eqs. (10.16) and (10.17) is the existence of fictitious forces in the rotating frame. Mathematically, the difference is simply the rearrangement of terms.

For cases in between the two special cases above, things aren't so clear, but that's the way it goes. Note that no matter what direction you move on a carousel, the Coriolis force always points in the same perpendicular direction relative to your motion. Whether it's to your right or to your left depends on the direction of the rotation. But given $\boldsymbol{\omega}$, you're stuck with the same relative direction of the force.

On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
'Twas like he was stalked
By some fiend always pushing him right.

Let's do some more examples . . .

Example (Dropped ball): A ball is dropped from height h, at a polar angle θ (measured down from the north pole). How far to the east is the ball deflected, by the time it hits the ground?

Solution: The angle between ω and $\mathbf{v}$ is $\pi - \theta$, so the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ is directed eastward with magnitude $2m\omega v \sin\theta$, where $v = gt$ is the speed at time

t (t runs from zero to the usual $\sqrt{2h/g}$).[1] Note that the ball is deflected to the east, independent of which hemisphere it is in. The eastward acceleration at time t is therefore $2\omega gt \sin\theta$. Integrating this to obtain the eastward speed (with an initial eastward speed of zero) gives $v_{\text{east}} = \omega g t^2 \sin\theta$. Integrating again to obtain the eastward deflection (with an initial eastward deflection of zero) gives $d_{\text{east}} = \omega g t^3 \sin\theta/3$. Plugging in $t = \sqrt{2h/g}$ gives

$$d_{\text{east}} = \frac{2\omega h \sin\theta}{3}\sqrt{\frac{2h}{g}}. \tag{10.18}$$

The frequency of the earth's rotation is $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$, so if we pick $\theta = \pi/2$ and $h = 100\,\text{m}$, for example, then we have $d_{\text{east}} \approx 2\,\text{cm}$.

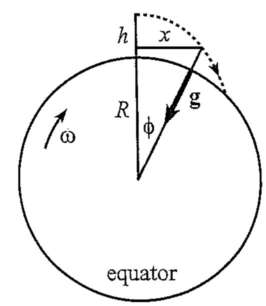

Fig. 10.8

REMARK: We can also solve this problem by working in an inertial frame; see Stirling (1983). Figure 10.8 shows the setup where a ball is dropped from a tower of height h located at the equator (the view is from the south pole). The earth is rotating in the inertial frame, so the initial sideways speed of the ball, $(R+h)\omega$, is larger than the sideways speed of the base of the tower, $R\omega$. This is the basic cause of the eastward deflection.

However, after the ball has moved to the right, the gravitational force on it picks up a component pointing to the left, and this slows down the sideways speed. If the ball has moved a distance x to the right, then the leftward component of gravity equals $g\sin\phi \approx g(x/R)$. Now, to leading order we have $x = R\omega t$, so the sideways acceleration of the ball is $a = -g(R\omega t/R) = -\omega g t$. Integrating this, and using the initial speed of $(R+h)\omega$, gives a rightward speed of $(R+h)\omega - \omega g t^2/2$. Integrating again gives a rightward distance of $(R+h)\omega t - \omega g t^3/6$. Subtracting off the rightward position of the base of the tower (namely $R\omega t$), and using $t \approx \sqrt{2h/g}$ (neglecting higher-order effects such as the curvature of the earth and the variation of g with altitude), we obtain an eastward deflection of $\omega h\sqrt{2h/g}(1 - 1/3) = (2/3)\omega h\sqrt{2h/g}$, relative to the base of the tower. If the ball is dropped at a polar angle θ instead of at the equator, then the only modification is that all speeds are decreased by a factor of $\sin\theta$, so we obtain the result in Eq. (10.18). ♣

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With Problems and Solutions

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