We want to write down the Lagrangian for a single particle in terms of coordinates $\boldsymbol{r}$ and $\dot{\boldsymbol{r}} \equiv \boldsymbol{v}$ that are measured w.r.t. a rotating frame (having rotational velocity vector $\boldsymbol{\Omega}$ w.r.t. the inertial frame, but same origin as inertial frame) and see that the expected pseudoforces emerge from the equation of motion.

To write the kinetic energy w.r.t. the inertial (non-rotating) frame, we use

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{space}} \;=\; \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} \;+\; \boldsymbol{\Omega}\times\boldsymbol{r}$$

which gives us

$$\boldsymbol{v}_o \;=\; \boldsymbol{v} + \Omega\times\boldsymbol{r} \;=\; \dot{\boldsymbol{r}} + \Omega\times\boldsymbol{r}$$

where $\boldsymbol{v}_o$ is the velocity in the inertial ("space") frame, while $\boldsymbol{v}$ is the velocity in the rotating ("body") frame. Now we can use $\boldsymbol{v}_o$ to write the KE w.r.t. the inertial frame:

$$\mathcal{L} \;=\; \frac{1}{2}m|\boldsymbol{v}_o|^2 - U(\boldsymbol{r}) \;=\; \frac{1}{2}m|\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r}|^2 - U(\boldsymbol{r})$$

Writing out the KE component by component (i=x,y,z):

$$\mathcal{L} \;=\; \left[\sum_i \frac{m}{2}(\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r})_i^2\right] - U(\boldsymbol{r})$$

Now pick coordinate n and differentiate $\mathcal{L}$. As usual, $\partial A^2/\partial r_n = (2A)(\partial A/\partial r_n)$.

$$\frac{\partial\mathcal{L}}{\partial r_n} = \left[\sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r})_i \frac{\partial}{\partial r_n}(\dot{r}_i + (\boldsymbol{\Omega}\times\boldsymbol{r})_i)\right] - \frac{\partial U(\boldsymbol{r})}{\partial r_n}$$

where I rewrote $(\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r})_i$ as $\dot{r}_i + (\boldsymbol{\Omega}\times\boldsymbol{r})_i$. The derivative of the first term is zero: $\partial\dot{r}_i/\partial r_n = 0$. We can write out the second term using the Cartesian Einstein notation as

$$(\boldsymbol{\Omega}\times\boldsymbol{r})_i = \sum_{jk}\Omega_j r_k \epsilon_{ijk}$$

whose derivative is

$$\frac{\partial}{\partial r_n}(\boldsymbol{\Omega}\times\boldsymbol{r})_i = \sum_{jk}\Omega_j\left(\frac{\partial r_k}{\partial r_n}\right)\epsilon_{ijk} = \sum_{jk}\Omega_j\delta_{kn}\epsilon_{ijk} = \sum_j \Omega_j\epsilon_{ijn}$$

Now we can plug this in to $\partial\mathcal{L}/\partial r_n$

$$\frac{\partial\mathcal{L}}{\partial r_n} = \left[\sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r})_i\left(\sum_j \Omega_j\epsilon_{ijn}\right)\right] - (\boldsymbol{\nabla}U)_n$$

$$\frac{\partial \mathcal{L}}{\partial r_n} = \left[\sum_{ij} m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \, \Omega_j \epsilon_{ijn} \right] - (\boldsymbol{\nabla} U)_n$$

Then using $\sum_{ij} A_i B_j \epsilon_{ijn} = (\boldsymbol{A} \times \boldsymbol{B})_n$ we rewrite this as a cross-product:

$$\frac{\partial \mathcal{L}}{\partial r_n} = m \left[(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} \right]_n - (\boldsymbol{\nabla} U)_n$$

and then distribute

$$\frac{\partial \mathcal{L}}{\partial r_n} = m(\boldsymbol{v} \times \Omega)_n + m((\boldsymbol{\Omega} \times \boldsymbol{r}) \times \Omega)_n - (\boldsymbol{\nabla} U)_n$$

Now let's go back and differentiate $\mathcal{L}$ w.r.t. $\dot{r}_n$ (dropping the potential term since $\partial U / \partial \dot{r}_n = 0$)

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_n} = \sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \, \frac{\partial}{\partial \dot{r}_n} (\dot{r}_i + (\boldsymbol{\Omega} \times \boldsymbol{r})_i)$$

Then use $\partial \dot{r}_i / \partial \dot{r}_n = \delta_{in}$ and $\partial r_i / \partial \dot{r}_n = 0$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_n} = \sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \, \delta_{in} = m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_n = m v_n + m(\boldsymbol{\Omega} \times \boldsymbol{r})_n$$

Now take the time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{r}_n} = m a_n + m(\dot{\boldsymbol{\Omega}} \times \boldsymbol{r})_n + m(\boldsymbol{\Omega} \times \boldsymbol{v})_n$$

So the Lagrange equation of motion for component r_n reads

$$m a_n + m(\dot{\boldsymbol{\Omega}} \times \boldsymbol{r})_n + m(\boldsymbol{\Omega} \times \boldsymbol{v})_n = m(\boldsymbol{v} \times \Omega)_n + m((\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega})_n - (\boldsymbol{\nabla} U)_n$$

Combining the components into vectors,

$$m\boldsymbol{a} + m\dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + m\boldsymbol{\Omega} \times \boldsymbol{v} = m\boldsymbol{v} \times \Omega + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} - \boldsymbol{\nabla} U$$

Then permute to flip signs, and use $\boldsymbol{F} = -\boldsymbol{\nabla} U$

$$m\boldsymbol{a} - m\boldsymbol{r} \times \dot{\boldsymbol{\Omega}} - m\boldsymbol{v} \times \boldsymbol{\Omega} = m\boldsymbol{v} \times \Omega + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} + \boldsymbol{F}$$

and rearrange to get vector sum of real force and the three pseudo-forces: "azimuthal" (a.k.a. Euler) force, Coriolis force, and centrifugal force.

$$m\boldsymbol{a} = \boldsymbol{F} + m\,\boldsymbol{r} \times \dot{\boldsymbol{\Omega}} + 2m\,\boldsymbol{v} \times \boldsymbol{\Omega} + m\,(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$