Physics 351, Spring 2015, Final Exam.

This closed-book exam has (only) 25% weight in your course grade. You can use one sheet of your own hand-written notes. Please show your work on these pages. The back side of each page is blank, so you can continue your work on the reverse side if you run out of space. Try to work in a way that makes your reasoning obvious to me, so that I can give you credit for correct reasoning even in cases where you might have made a careless error. Correct answers without clear reasoning may not receive full credit. Clear reasoning is especially important for "show that" questions.

The last page of the exam contains a list of equations that you might find helpful, to complement your own sheet of notes. You can detach it now if you like, before we begin.

The exam contains five questions, of equal weight. So each question is worth 20%. You might want to start with whichever questions you find easiest.

Because I believe that most of the learning in a physics course comes from your investing the time to work through homework problems, most of these exam problems are similar or identical to problems that you have already solved. The only point of the exams, in my opinion, is to motivate you to take the weekly homework seriously. So you should find nothing surprising in this exam.

Name: Bill

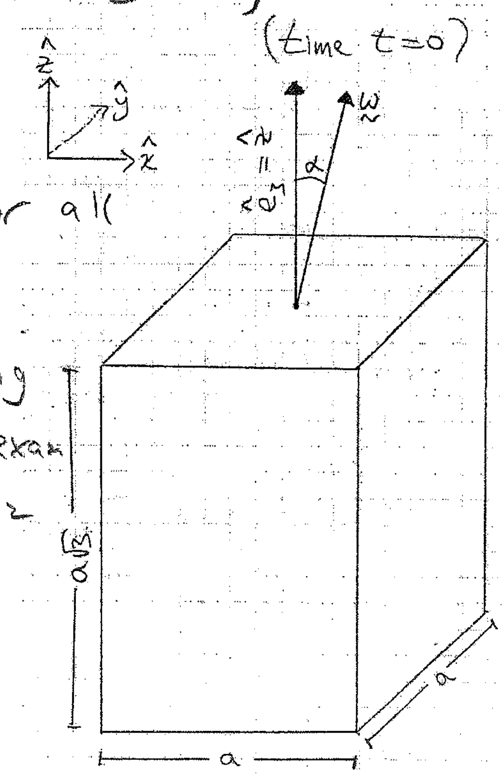
Problem 1.

A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\,a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{\mathbf{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\mathbf{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\mathbf{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\,\hat{\boldsymbol{z}} + \sin\alpha\,\hat{\boldsymbol{x}})$.

(about the CoM)

(a) Show (or argue) that the inertia tensor has the form

$$\underline{\underline{I}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and find the constant } I_0.$$

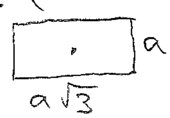


$I_{xy} = -\int xy\,dm = 0$ by symmetry, as for all off-diagonal elements.

$I_{zz} = \int (x^2+y^2)\,dm = \frac{m}{12}(a^2+a^2) = \frac{1}{6}ma^2$ using result for flat plate from back page of exam

$I_{xx} = \int (y^2+z^2)\,dm = \frac{m}{12}\left(a^2+(\sqrt{3}\,a)^2\right) = \frac{1}{3}ma^2$

using flat plate



So $\boxed{I_0 = \frac{1}{6}ma^2}$

$\lambda_1 = \lambda_2 = 2I_0\,, \quad \lambda_3 = I_0$

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\mathbf{e}}_1$,$\hat{\mathbf{e}}_2$,$\hat{\mathbf{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

$\underset{\sim}{L} = \underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda_1\omega_1\hat{e}_1 + \lambda_3\omega_3\hat{e}_3 = 2I_0\omega\sin\alpha\,\hat{e}_1 + I_0\omega\cos\alpha\,\hat{e}_3$

since at $t=0$ $\hat{e}_1 = \hat{x}$, $\hat{e}_3 = \hat{z}$, $\underset{\sim}{\omega} = \omega\sin\alpha\,\hat{x} + \omega\cos\alpha\,\hat{z}$.

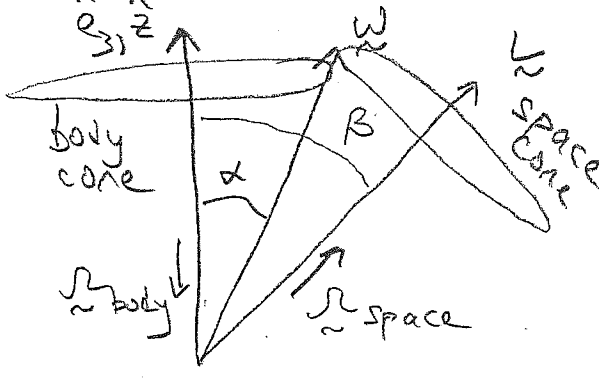
We can also write $\underset{\sim}{L} = 2I_0\omega\sin\alpha\,\hat{x} + I_0\omega\cos\alpha\,\hat{z}$.

This second expression remains true for $t>0$, because

$0 = \underset{\sim}{\tau} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{\text{space frame}} \Rightarrow \underset{\sim}{L}$ is constant in space frame for torque-free rotation.

The unit vectors $\hat{e}_i$ will rotate with the body. In particular, we will see L_1 and L_2 are not constant.

(c) Draw a sketch showing the vectors $\hat{\mathbf{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).



$$\tan\alpha = \frac{\omega_1}{\omega_3} \qquad \tan\beta = \frac{L_1}{L_3} = 2\tan\alpha$$

body cone is traced out by $\underset{\sim}{\omega}$ as it precesses about $\hat{e}_3$ in body frame.
space cone is traced out by $\underset{\sim}{\omega}$ as it precesses about $\underset{\sim}{L}$ in space frame.

Note that $\underset{\sim}{L}$, $\underset{\sim}{\omega}$, $\hat{e}_3$ remain coplanar:

$$\left.\begin{aligned} \hat{e}_3 &= 0\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \hat{e}_3 \\ \underset{\sim}{L} &= \lambda_1\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \lambda_3\omega_3\hat{e}_3 \\ \underset{\sim}{\omega} &= (\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \omega_3\hat{e}_3 \end{aligned}\right\}$$

coplanar: plane defined by $\hat{e}_3$ and $\omega_1\hat{e}_1 + \omega_2\hat{e}_2 = \vec{\omega}_\perp$

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\mathbf{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\mathbf{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\mathbf{body}}$.

$$\underset{\sim}{\Omega}_{space} = \frac{\underset{\sim}{L}}{\lambda_1} = \omega\sin\alpha\,\hat{x} + \frac{\lambda_3}{\lambda_1}\omega\cos\alpha\,\hat{z} = \omega(\sin\alpha\,\hat{x} + \tfrac{1}{2}\cos\alpha\,\hat{z})$$

$$= \left(\frac{\omega}{2}\sqrt{\cos^2\alpha + 4\sin^2\alpha}\right)\hat{L} = \left(\frac{\omega}{2}\sqrt{1+3\sin^2\alpha}\right)\hat{L}$$

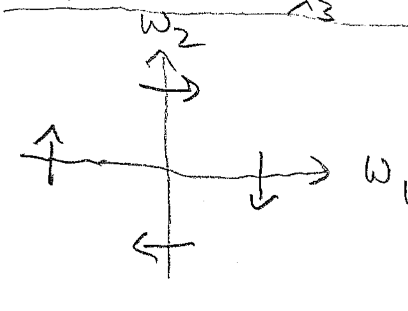
At $t=0$, $\underset{\sim}{\Omega}_{space} = \omega\sin\alpha\,\hat{e}_1 + \frac{1}{2}\omega\cos\alpha\,\hat{e}_3$

$$\underset{\sim}{\Omega}_{body} = \frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\hat{e}_3 = \frac{I_0 - 2I_0}{2I_0}\omega_3\hat{e}_3 = -\frac{1}{2}\omega\cos\alpha\,\hat{e}_3$$

$$\boxed{\dot{\omega}_3 = \omega_1\omega_2\frac{\lambda_1-\lambda_2}{\lambda_3} = 0}$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1} = \frac{1}{2}\omega_3\omega_2 = \left(\tfrac{1}{2}\omega\cos\alpha\right)\omega_2$$

$$\dot{\omega}_2 = \omega_3\omega_1\frac{\lambda_3-\lambda_1}{\lambda_2} = -\frac{1}{2}\omega_3\omega_1 = -\left(\tfrac{1}{2}\omega\cos\alpha\right)\omega_1$$



clockwise $\rightarrow$ $\underset{\sim}{\omega}$ precesses in $-\hat{e}_3$ direction in body frame

(f) You argued in HW11 that $\mathbf{\Omega_{space}} = \mathbf{\Omega_{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\mathbf{\Omega_{space}}$ and $\mathbf{\Omega_{body}}$ that you calculate for $t = 0$.

At $t=0$, $\underset{\sim}{\Omega}_{body} = -\frac{1}{2}\omega\cos\alpha\,\hat{e}_3$

$\underset{\sim}{\omega} = \omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

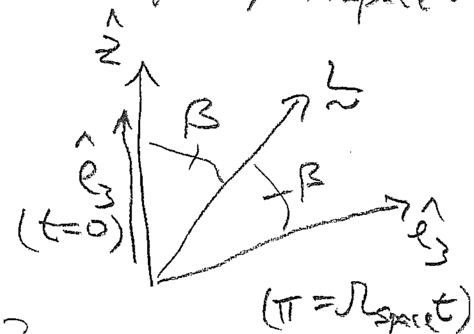
add to $+\frac{1}{2}\omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

At $t=0$, $\underset{\sim}{\Omega}_{space} = \omega\sin\alpha\,\hat{e}_1 + \frac{1}{2}\omega\cos\alpha\,\hat{e}_3$ ✓

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

The initial angle between $\hat{e}_3$ and $\underset{\sim}{L}$ is $\beta = \operatorname{atan}(2\tan\alpha) \simeq 2\alpha$. This angle is a constant of the motion, because $L_3 = \lambda_3\omega_3 = \text{const.}$ and $\underset{\sim}{L}_{space}$ (hence magnitude of $\underset{\sim}{L}_{body}$) is constant. As shown on diagram (c), $\underset{\sim}{L}, \underset{\sim}{\omega}, \hat{e}_3$ remain coplanar. So in the space frame, $\underset{\sim}{\omega}$ and $\hat{e}_3$ both precess about $\underset{\sim}{L}$ with frequency Ω_{space}.

Maximum angle between $\hat{e}_3$ and $\hat{z}$ is $\boxed{2\beta \simeq 4\alpha}$.



(h) At what time t is this maximum deviation first reached?

one-half period of Ω_{space}:

$\Omega_{space}t = \pi$

$$t = \frac{\pi}{\Omega_{space}} = \frac{2\pi}{\omega\sqrt{1+3\sin^2\alpha}} \simeq \frac{2\pi}{\omega} \quad \left(\simeq \frac{2\pi}{\omega_3} \text{ if } \alpha \ll 1\right)$$

So the precession ("wobble") has $\simeq$ half the frequency of the "spin" if $\alpha \ll 1$.

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

Problem 2. *Hamiltonian treatment of the heavy symmetric top.*
Consider a symmetric top ($\lambda_1 = \lambda_2$) whose tip has a fixed location in space. Using the Euler angles ϕ, θ, and ψ (whose detailed definitions are not needed for you to solve this problem) to represent the top's orientation, the top's Lagrangian can be written as

$$\mathcal{L} = \underbrace{\boxed{\frac{1}{2}\lambda_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2}}_{T} - \underbrace{\boxed{MgR\cos\theta}}_{U}$$

where M is the mass of the top and R is the distance from the contact point to the top's center of mass. λ_3 is the moment of inertia for the top's symmetry axis, and λ_1 is the moment of inertia for the other two principal axes.

(a) Calculate the three generalized momenta, p_ϕ, p_θ, and p_ψ.

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \lambda_1 \dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \lambda_1\dot{\theta} \rightarrow P_\theta^2 = \lambda_1^2\dot{\theta}^2 \rightarrow \frac{P_\theta^2}{2\lambda_1} = \frac{1}{2}\lambda_1\dot{\theta}^2$$

$$P_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) \rightarrow \frac{P_\psi^2}{2\lambda_3} = \frac{\lambda_3^2}{2\lambda_3}(\dot{\psi} + \dot{\phi}\cos\theta)^2 = \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$(P_\phi - P_\psi\cos\theta)^2 = \lambda_1^2\dot{\phi}^2\sin^4\theta \rightarrow \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} = \frac{1}{2}\lambda_1\dot{\phi}^2\sin^2\theta$$

(b) The simplest way to construct the Hamiltonian is to realize that the coordinates are natural, so $\mathcal{H} = T + U$. Use this to show that the Hamiltonian is given by

$\mathcal{L} = T - U$ $\qquad U = MgR\cos\theta$

$$\mathcal{H} = \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3} + MgR\cos\theta$$

$$T = \frac{1}{2}\lambda_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$T = \left(\frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta}\right) + (P_\theta^2/2\lambda_1) + (P_\psi^2/2\lambda_3)$$

$$\mathcal{H} = T + U = \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{P_\theta^2}{2\lambda_1} + \frac{P_\psi^2}{2\lambda_3} + MgR\cos\theta \quad ✓$$

If I write the position of some speck of the top's mass as $a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3$, then I can recast this as $A\hat{x} + B\hat{y} + C\hat{z}$ where $A = A(\phi,\theta,\psi)$, $B = B(\phi,\theta,\psi)$, etc., with no explicit time dependence. So $\mathcal{H} = T + U$ if I write T, U in terms of Euler angles ϕ, θ, ψ.

(c) Two of the Euler-angle coordinates are ignorable. Which ones? The corresponding generalized momenta are constant. Use this to show that the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_\theta^2}{2\lambda_1} + U_{\text{eff}}(\theta)$$

where $U_{\text{eff}}(\theta)$ depends only on θ and on constants of the motion. What is the effective potential energy U_{eff} for this system?

ϕ and ψ do not appear in the Lagrangian, so are ignorable.

$\Rightarrow P_\phi \equiv \text{const.} \equiv L_z$, $P_\psi \equiv \text{const.} \equiv L_3$ So $\mathcal{H} = \frac{p_\theta^2}{2\lambda_1} + U_{\text{eff}}(\theta)$

with $U_{\text{eff}}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$, which depends only on θ and on constants of the motion.

(d) Write Hamilton's two equations of motion for θ. (Don't bother to evaluate $dU_{\text{eff}}/d\theta$.) Combine these to find $\ddot{\theta}$ in terms of $dU_{\text{eff}}/d\theta$.

$$\dot{P}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{dU_{\text{eff}}}{d\theta} \qquad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial P_\theta} = \frac{P_\theta}{\lambda_1}$$

$$\rightarrow \ddot{\theta} = \frac{\dot{P}_\theta}{\lambda_1} = \boxed{-\frac{1}{\lambda_1}\frac{dU_{\text{eff}}}{d\theta} = \ddot{\theta}}$$

(e) If θ oscillates back and forth between θ_{min} and θ_{max}, this periodic motion of θ is called (circle one): (i) precession, (ii) rotation, (iii) nutation, (iv) vibration, (v) revolution.

(f) In general (not directly related to the above problem), what must be true of the relationship between the inertial Cartesian coordinates (e.g. x, y) and the generalized coordinates (e.g. q_1, q_2) for the Hamiltonian $\mathcal{H}(q_1, q_2, p_1, p_2, t)$ to equal the total energy of the system?

$x = x(q_1, q_2)$, not $x = x(q_1, q_2, t)$

$y = y(q_1, q_2)$, not $y = y(q_1, q_2, t)$

For example, $x = q_1 \cos q_2$, $y = q_1 \sin q_2$. But not $x = q_1 + vt$.

(g) If $\partial \mathcal{L}/\partial t = 0$, what statement can be made about $\mathcal{H}$?

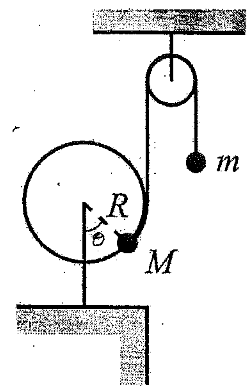
Then $\frac{\partial \mathcal{H}}{\partial t} = 0 \Rightarrow \frac{d\mathcal{H}}{dt} = 0 \Rightarrow \mathcal{H}$ is a constant of the motion.

$$\frac{d\mathcal{H}}{dt} = \underbrace{\frac{\partial \mathcal{H}}{\partial q}}_{-\dot{p}}\dot{q} + \underbrace{\frac{\partial \mathcal{H}}{\partial p}}_{\dot{q}}\dot{p} + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

$$\boxed{\text{So } \frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \frac{d\mathcal{H}}{dt} = 0.}$$

Problem 3.

A mass M is attached to a massless hoop of radius R that lies in a vertical plane and is free to rotate about its fixed center. M is tied to a string that winds part way around the hoop and then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string. Assume that m moves only vertically, that $M > m$, and that $\theta \leq \pi/2$, so that the vertical displacement of m is given by $R\theta$.



(a) Find the Lagrange equation of motion for the angle θ of rotation of the hoop, where $\theta = 0$ would put M directly below the center of the hoop.

(b) Find an expression for the equilibrium angle θ_0.

(c) What is the frequency of small oscillations about the equilibrium angle θ_0?

$$U = -MgR\cos\theta - mgR\theta \qquad T = \tfrac{1}{2}(m+M)R^2\dot\theta^2$$

$$\mathcal{L} = \tfrac{1}{2}(m+M)R^2\dot\theta^2 + MgR\cos\theta + mgR\theta$$

(a) $$\frac{\partial \mathcal{L}}{\partial \theta} = -MgR\sin\theta + mgR = gR(m - M\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{d}{dt}\left((m+M)R^2\dot\theta\right) = \boxed{(m+M)R^2\ddot\theta = gR(m - M\sin\theta)}$$

(b) $$\ddot\theta = 0 \Rightarrow m = M\sin\theta_0 \rightarrow \boxed{\sin\theta_0 = m/M}$$

(c) $$\ddot\theta = \frac{g}{(m+M)R}(m - M\sin\theta) \equiv f(\theta) \rightarrow f'(\theta) = -\frac{Mg\cos\theta}{(m+M)R}$$

$$\theta = \theta_0 + \varepsilon \rightarrow \ddot\varepsilon = \ddot\theta = f(\theta) = f(\theta_0 + \varepsilon) \simeq \underbrace{\cancel{f(\theta_0)}}_{0 \text{ by construction of } \theta_0} + \varepsilon f'(\theta_0)$$

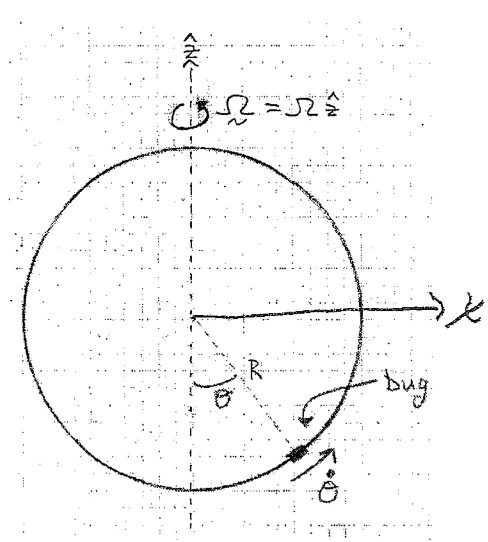
$$\ddot\varepsilon = -\varepsilon\frac{Mg\cos\theta_0}{(m+M)R} = -\varepsilon\frac{Mg}{(m+M)R}\sqrt{1 - \left(\frac{m}{M}\right)^2} = -\varepsilon\frac{\cancel{M}g}{(m+M)R}\sqrt{\frac{M^2 - m^2}{\cancel{M^2}}}$$

$$\ddot\varepsilon = -\varepsilon\frac{g}{(m+M)R}\sqrt{(M+m)(M-m)} = -\varepsilon\frac{g}{R}\sqrt{\frac{M-m}{M+m}} = \ddot\varepsilon$$

$$\ddot\varepsilon = -\omega^2\varepsilon \Rightarrow \omega^2 = \frac{Mg\cos\theta_0}{(m+M)R} = \frac{g}{R}\sqrt{\frac{M-m}{M+m}}$$

$$\omega = \sqrt{\frac{Mg\cos\theta_0}{(M+m)R}} = \sqrt{\frac{g}{R}}\left(\frac{M-m}{M+m}\right)^{\frac{1}{4}}$$

Problem 4.
A hoop of radius R is (by some external mechanism) made to rotate at constant angular velocity $\boldsymbol{\Omega} = \Omega\,\hat{z}$ around a diameter, which coincides with the z axis. A small bug of mass m walks at constant angular speed $\dot{\theta} \equiv \omega$ around the hoop. Let $\boldsymbol{F}$ be the total force that the hoop applies to the bug when the bug is at the angle θ shown, and let $F_\perp$ be the component of $\boldsymbol{F}$ that is perpendicular to the plane of the hoop. At the instant depicted in the figure, $F_\perp$ points into the plane of the page. Ignore gravity in this problem. Find $F_\perp$ in two ways, whose results should agree:



(a) Work in the lab frame: at the angle θ, find the rate of change of L_z, the z component of the bug's angular momentum (i.e. the angular momentum about the hoop's rotation axis), and then consider the torque component τ_z on the bug. Use τ_z to find $F_\perp$.

$$L_z = I\Omega \Rightarrow m\,(R\sin\theta)^2\,\Omega$$

$$\tau_z = \frac{dL_z}{dt} = mR^2\Omega \cdot 2\sin\theta\cos\theta\,\dot{\theta} = 2mR^2\Omega\omega\sin\theta\cos\theta$$

Since $F_\perp$ points into the page,

$$\tau_z = R_x F_\perp = (R\sin\theta)F_\perp \Rightarrow F_\perp = \frac{\tau_z}{R\sin\theta} = \boxed{2mR\Omega\omega\cos\theta}$$

(See next page for complicated alternate solution.)

(b) Work in the rotating frame of the hoop: at the angle θ, find the relevant fictitious force, and then argue that this fictitous force must be balanced by $F_\perp$ (exerted by the hoop), since the bug remains within the plane of the hoop. (The relevant fictitious force depends on how quickly the bug is moving (in the frame of the hoop) and points in a direction that is perpendicular to the bug's motion.)

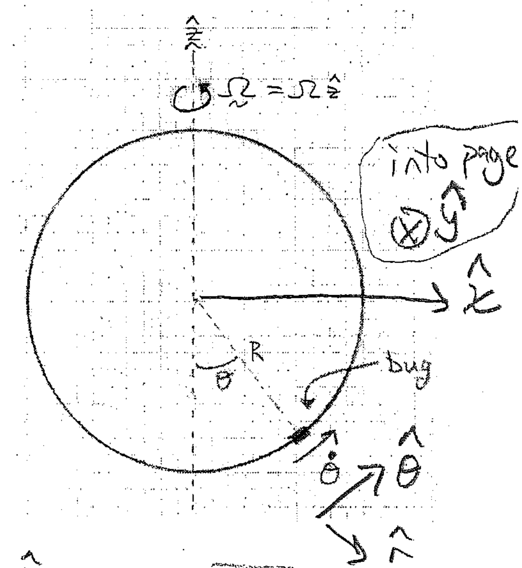
The bug does not leave the plane of the hoop. The coriolis force is $2m\underset{\sim}{v} \times \underset{\sim}{\Omega} = 2m(R\dot{\theta}\hat{\theta}) \times (\Omega\hat{z})$

$= 2mR\dot{\theta}\Omega\cos\theta$ (out of plane of hoop). (The centrifugal force is within the plane of the hoop.)

This coriolis force must be balanced by component $F_\perp$ of the contact force exerted by hoop on bug.

So $\boxed{F_\perp = 2mR\Omega\omega\cos\theta}$ and points into plane of the page.

Problem 4.
A hoop of radius R is (by some external mechanism) made to rotate at constant angular velocity $\boldsymbol{\Omega} = \Omega\,\hat{\boldsymbol{z}}$ around a diameter, which coincides with the z axis. A small bug of mass m walks at constant angular speed $\dot{\theta} \equiv \omega$ around the hoop. Let $\boldsymbol{F}$ be the total force that the hoop applies to the bug when the bug is at the angle θ shown, and let $F_\perp$ be the component of $\boldsymbol{F}$ that is perpendicular to the plane of the hoop. At the instant depicted in the figure, $F_\perp$ points into the plane of the page. Ignore gravity in this problem. Find $F_\perp$ in two ways, whose results should agree:



(a) Work in the lab frame: at the angle θ, find the rate of change of L_z, the z component of the bug's angular momentum (i.e. the angular momentum about the hoop's rotation axis), and then consider the torque component τ_z on the bug. Use τ_z to find $F_\perp$.

Complicated alternate solution:

Let $\hat{y}$ point into plane of hoop, so that $\hat{x} \times \hat{y} = \hat{z}$.

$$\hat{r} \times \hat{\theta} = -\hat{y}$$
$$\hat{r} \times \hat{y} = \hat{\theta}$$

Bug's velocity $\underset{\sim}{v} = R\dot{\theta}\,\hat{\theta} + R\sin\theta\,\Omega\,\hat{y}$

$$\underset{\sim}{L} = \underset{\sim}{r} \times m\underset{\sim}{v} = R\hat{r} \times m(R\dot{\theta}\hat{\theta} + R\sin\theta\,\Omega\hat{y}) = mR^2(-\dot{\theta}\hat{y} + \Omega\sin\theta\,\hat{\theta})$$

$$\underset{\sim}{\tau} = \dot{\underset{\sim}{L}} = mR^2\left(-\dot{\theta}\frac{d\hat{y}}{dt} + \Omega\cos\theta\,\dot{\theta}\,\hat{\theta} + \Omega\sin\theta\frac{d\hat{\theta}}{dt}\right) \leftarrow \text{note } \ddot{\theta} = 0)$$

$$= mR^2\left(\dot{\theta}\Omega(\hat{\theta}\cos\theta + \hat{r}\sin\theta) + \Omega\dot{\theta}\cos\theta\,\hat{\theta} + \Omega\sin\theta(-\dot{\theta}\hat{r} + \Omega\cos\theta\,\hat{y})\right)$$

$$= mR^2(2\Omega\dot{\theta}\cos\theta\,\hat{\theta} + \Omega^2\sin\theta\cos\theta\,\hat{y}) = R\hat{r} \times \underset{\sim}{F} = R\hat{r} \times (F_\perp\hat{y} + F_r\hat{r} + F_\theta\hat{\theta})$$

$$= RF_\perp\hat{\theta} + 0 - RF_\theta\hat{y} \Rightarrow \boxed{F_\perp = 2mR\Omega\dot{\theta}\cos\theta}$$

we want the component of $\underset{\sim}{F}$ that is $\perp$ to plane of hoop

(b) Work in the rotating frame of the hoop: at the angle θ, find the relevant fictitious force, and then argue that this fictitous force must be balanced by $F_\perp$ (exerted by the hoop), since the bug remains within the plane of the hoop. (The relevant fictitious force depends on how quickly the bug is moving (in the frame of the hoop) and points in a direction that is perpendicular to the bug's motion.)

using $$\frac{d\hat{y}}{dt} = -\Omega\hat{x} = -\Omega(\hat{\theta}\cos\theta + \hat{r}\sin\theta)$$

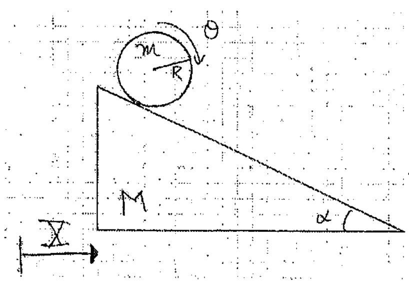
$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} + \Omega\cos\theta\,\hat{y}$$

Several of you solved part (a) this way, so I include it here. I recommend the simpler approach, though this vector manipulation can be fun if you're not under time pressure.

Problem 5.

Anecdote: My high-school physics teacher, Mr. Rodriguez, once wrote this problem on the board as a challenge. He told us that only one of his students had ever succeeded at solving it (using Newtonian methods, of course): that student's dad was the famous MIT linguistics professor Noam Chomsky.

One of my high-school best friends and I both studied Analytical Mechanics as college juniors (at different colleges). While sharing college stories over winter break, my friend and I were simultaneously inspired to solve Mr. Rodriguez's "impossible" problem using the Lagrangian technique we had just learned. We worked it out side-by-side on the board, and agreed on the final answer. Amazing! Now you too can quickly work through this problem that would have been quite difficult before you learned Lagrangian mechanics.



A uniform solid cylinder ($I = \frac{1}{2}mR^2$) of mass m and radius R rolls without slipping on the inclined surface of a wedge of mass M. The inclined surface of the wedge makes an angle α w.r.t. the horizontal. The wedge slides on a frictionless horizontal surface. Let θ be the angle through which the cylinder has rolled (clockwise, downhill) about its axis. Let X be the horizontal displacement (to the right) of the wedge. With our sign convention, if $\ddot{\theta} > 0$, we expect $\ddot{X} < 0$.

(a) Write the Lagrangian using generalized coordinates θ and X. $U = -mg(R\theta)\sin\alpha$

$$T = \tfrac{1}{2}\left(\tfrac{1}{2}mR^2\right)\dot{\theta}^2 + \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(R\dot{\theta}\sin\alpha)^2 + \tfrac{1}{2}m(R\dot{\theta}\cos\alpha + \dot{X})^2$$

$$= \tfrac{1}{4}mR^2\dot{\theta}^2 + \tfrac{1}{2}(m+M)\dot{X}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 + mR\dot{\theta}\dot{X}\cos\alpha$$

$$\mathcal{L} = T - U = \tfrac{3}{4}mR^2\dot{\theta}^2 + \tfrac{1}{2}(m+M)\dot{X}^2 + mR\dot{\theta}\dot{X}\cos\alpha + mgR\theta\sin\alpha$$

(b) Write the Lagrange equation for X.

$$\frac{\partial \mathcal{L}}{\partial X} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) = \frac{d}{dt}\left[\underbrace{(m+M)\dot{X} + mR\dot{\theta}\cos\alpha}_{\text{constant of the motion}}\right]$$

$$\text{or} \quad (m+M)\ddot{X} + mR\ddot{\theta}\cos\alpha = 0$$

$$\longrightarrow \quad \ddot{X} = -\frac{mR\cos\alpha}{m+M}\,\ddot{\theta}$$

(c) Interpret your answer for (b) in terms of a well-known conservation principle.

X is "ignorable" (does not appear in $\mathcal{L}$) $\rightarrow P_X$ is a constant of the motion.

P_X is horizontal linear momentum of wedge + cylinder system. P_X is constant because there are no external horizontal forces.

conservation of linear momentum

(d) Write the Lagrange equation for θ.

$$\frac{\partial \mathcal{L}}{\partial \theta} = mgR\sin\alpha \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{d}{dt}\left[\frac{3}{2}mR^2\dot\theta + mR\dot{x}\cos\alpha\right]$$

$$mgR\sin\alpha = \frac{3}{2}mR^2\ddot\theta + mR\ddot{x}\cos\alpha$$

$$g\sin\alpha = \frac{3}{2}R\ddot\theta + \ddot{x}\cos\alpha$$

from (b), $\ddot{X} = -\dfrac{mR\cos\alpha}{m+M}\ddot\theta$

(e) Derive an expression for $\ddot\theta$ by using (b) to eliminate $\ddot X$ from (d). (This expression is the answer to Mr. Rodriguez's problem.)

$$\frac{3}{2}mR^2\ddot\theta - (mR\cos\alpha)\left(\frac{mR\cos\alpha}{m+M}\right)\ddot\theta = mgR\sin\alpha$$

$$\ddot\theta\left(\frac{3}{2}mR^2 - \frac{m^2R^2\cos^2\alpha}{m+M}\right) = mgR\sin\alpha \rightarrow R\ddot\theta\left(\frac{3}{2} - \frac{m\cos^2\alpha}{m+M}\right) = g\sin\alpha$$

$$R\ddot\theta = \frac{g\sin\alpha}{\frac{3}{2} - \frac{m\cos^2\alpha}{m+M}}$$

$\rightarrow$ really $\frac{M}{m} \rightarrow \infty$ limit.

(f) As a check, show that in the $M \to \infty$ limit, your answer for (e) reduces to the "freshman physics" result (for an immovable wedge)

$$R\ddot\theta = \frac{g\sin\alpha}{1 + \frac{I}{mR^2}} = \frac{2}{3}g\sin\alpha$$

where $I = \frac{1}{2}mR^2$ for a uniform solid cylinder.

$$\ddot\theta = \frac{g\sin\alpha/R}{\frac{3}{2} - \frac{\cos^2\alpha}{1+\frac{M}{m}}} \longrightarrow \frac{g\sin\alpha/R}{\frac{3}{2}} = \frac{\frac{2}{3}g\sin\alpha}{R} \quad \text{as } \frac{M}{m} \to \infty$$

because $\dfrac{\cos^2\alpha}{1+\frac{M}{m}} \to 0$ as $\dfrac{M}{m} \to \infty$.

Possibly useful equations.

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega}\times\boldsymbol{Q}$$

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}}\times\boldsymbol{\Omega} + m\,(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega} = \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m\Omega^2\rho\,\hat{\boldsymbol{\rho}}$$

For a uniform solid cylinder of radius R about its symmetry axis, $I = mR^2/2$. For a uniform thin rod of length L about its center (perpendicular to the rod axis), $I = mL^2/12$. For a rectangular plate about its center (rotation axis normal to plate), $I = m(a^2+b^2)/12$, where a and b are the short and long side lengths.

For a free symmetric top, $\boldsymbol{\Omega}_s = \boldsymbol{L}/\lambda_1$

Euler equations:

$$\begin{aligned}
\tau_1 &= \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3\\
\tau_2 &= \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_3\omega_1\\
\tau_3 &= \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2
\end{aligned}$$

If $\boldsymbol{\tau} = 0$ and $\lambda_1 = \lambda_2$ then the Euler equations reduce to the simpler form

$$\dot{\omega}_3 = \frac{\lambda_1-\lambda_1}{\lambda_3}\omega_1\omega_2 = 0$$

$$\dot{\omega}_1 = \frac{\lambda_1-\lambda_3}{\lambda_1}\omega_2\omega_3 = -\left(\frac{\lambda_3-\lambda_1}{\lambda_1}\omega_3\right)\omega_2 = -\,\Omega_b\,\omega_2$$

$$\dot{\omega}_2 = \frac{\lambda_3-\lambda_1}{\lambda_1}\omega_3\omega_1 = \left(\frac{\lambda_3-\lambda_1}{\lambda_1}\omega_3\right)\omega_1 = \Omega_b\,\omega_1$$

so we can represent the precession of the $\boldsymbol{\omega}$ vector as an angular velocity vector $\boldsymbol{\Omega}_b$ with

$$\boldsymbol{\Omega}_b = \left(\frac{\lambda_3-\lambda_1}{\lambda_1}\omega_3\right)\hat{\boldsymbol{e}}_3.$$

Euler-angle convention: Start with body axes aligned with space axes. (i) Rotate body through angle ϕ about $\hat{\boldsymbol{z}}$. This leaves $\hat{\boldsymbol{e}}_3$ alone but rotates the first and second body axes in the xy plane. In particular, the second body axis now points in a direction called $\hat{\boldsymbol{e}}_2'$. (ii) Rotate body through angle θ about the new axis $\hat{\boldsymbol{e}}_2'$. This moves the body axis $\hat{\boldsymbol{e}}_3$ to the direction whose polar angles are θ and ϕ. (iii) Rotate the body about $\hat{\boldsymbol{e}}_3$ through whatever angle ψ is needed to bring the body axes $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_1$ into their assigned directions.

At any instant, you can use the values of ϕ,θ,ψ at that instant to write each body unit vector $\hat{\boldsymbol{e}}_i$ as a linear combination of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. The coefficients involve sines and cosines of ϕ,θ,ψ but have no explicit time dependence.