Physics 351, Spring 2017, Final Exam.

This closed-book exam has (only) 25% weight in your course grade. You can use one 3×5 card of your own hand-written notes. Please show your work on these pages. The back side of each page is blank, so you can continue your work on the reverse side if you run out of space. Try to work in a way that makes your reasoning obvious to me, so that I can give you credit for correct reasoning even in cases where you might have made a careless error. Correct answers without clear reasoning may not receive full credit.

The last page of the exam contains a list of equations that you might find helpful, to complement your own note card. You can detach it now if you like, before we begin.

The exam contains four questions, of equal weight. So each question is worth 25%. You might want to start with whichever questions you find easiest.

Because I believe that most of the learning in a physics course comes from your investing the time to work through homework problems, most of these exam problems are similar or identical to problems that you have already solved. The only point of the exams, in my opinion, is to motivate you to take the weekly homework seriously. So you should find nothing very surprising in this exam.

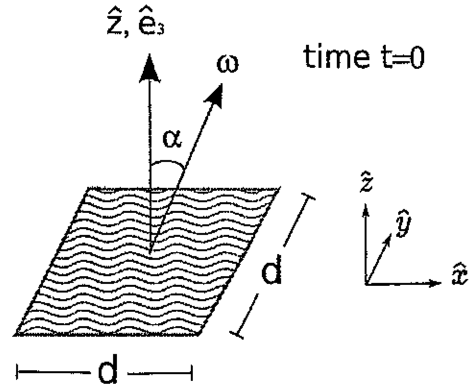
Name: BILL

Problem 1.

A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. (Imagine a dinner plate tossed in the air like a frisbee, but neglecting air resistance.) At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.

(a) Show (or argue) that the inertia tensor has the form

$$\underline{\underline{I}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ and find the constant I_0.



$\lambda_1 =$ same as thin rod : $\dfrac{md^2}{12}$

$\lambda_2 = \lambda_1$ by symmetry

$\lambda_3 = \lambda_1 + \lambda_2 = \dfrac{md^2}{6}$ because

$\lambda_3 = \lambda_1 + \lambda_2$ for a planar object

$\Rightarrow \boxed{I_0 = \dfrac{md^2}{12}}$ Because of reflection symmetry about x, y, z axes (at $t=0$), $\hat{x}, \hat{y}, \hat{z}$ are the principal axes (at $t=0$).

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\boldsymbol{e}}_1$,$\hat{\boldsymbol{e}}_2$,$\hat{\boldsymbol{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

$$\underset{\sim}{L}(t=0) = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3 = I_0 \omega (\sin\alpha \hat{e}_1 + 2\cos\alpha \hat{e}_3)$$

$$\underset{\sim}{L}(t=0) = I_0 \omega (\sin\alpha \hat{x} + 2\cos\alpha \hat{z})$$

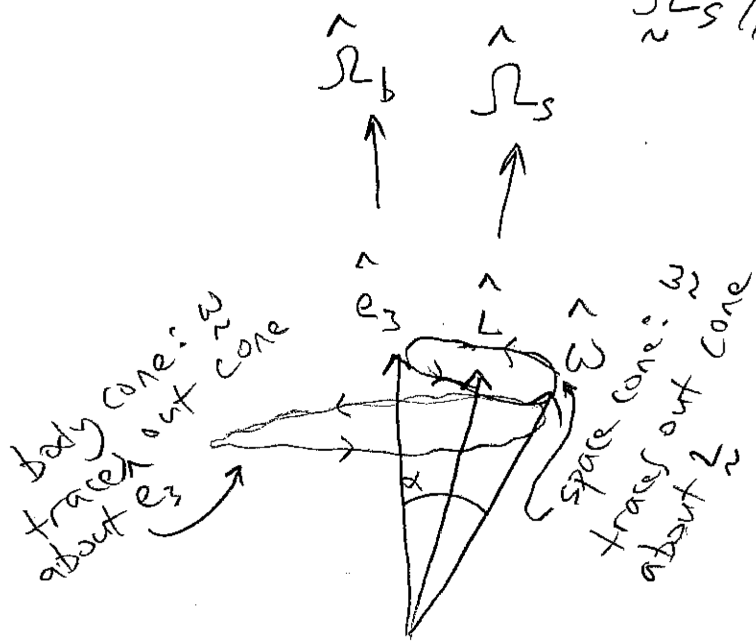
The second expression will continue to be valid for $t > 0$ because $\underset{\sim}{0} = \underset{\sim}{\tau} = \left(\dfrac{d\underset{\sim}{L}}{dt}\right)_{space}$ for torque-free rotation.

(c) Draw a sketch showing the vectors $\hat{e}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).

$\underset{\sim}{\Omega}_s \,//\, \underset{\sim}{L}$ always

$\underset{\sim}{\Omega}_b \,//\, \hat{e}_3$ for $\lambda_3 > \lambda_1$

(if $\lambda_3 < \lambda_1$ then $\underset{\sim}{\Omega}_b \,//\, -\hat{e}_3$)



(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\mathbf{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\mathbf{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\mathbf{body}}$.

$$\underset{\sim}{\Omega}_{body} = \frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3 \hat{e}_3 = \omega_3 \hat{e}_3 = \omega \cos\alpha\, \hat{e}_3$$

$$\underset{\sim}{\Omega}_{space} = \frac{\underset{\sim}{L}}{\lambda_1} = \omega\left(\sin\alpha\, \hat{x} + 2\cos\alpha\, \hat{z}\right)$$

(f) You argued in HW11 that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

$$\underset{\sim}{\Omega}_s = \omega(\sin\alpha\,\hat{x} + 2\cos\alpha\,\hat{z})$$

$$\underset{\sim}{\Omega}_b = \omega\cos\alpha\,\hat{e}_3$$

at $t=0$, $\hat{e}_3 = \hat{z}$, so $\underset{\sim}{\Omega}_b(t=0) = \omega\cos\alpha\,\hat{z}$

$$\left[\underset{\sim}{\Omega}_b + \underset{\sim}{\omega}\right]_{t=0} = (\omega\cos\alpha\,\hat{z}) + (\omega\sin\alpha\,\hat{x} + \omega\cos\alpha\,\hat{z})$$

$$= \omega\sin\alpha\,\hat{x} + 2\omega\cos\alpha\,\hat{z}$$

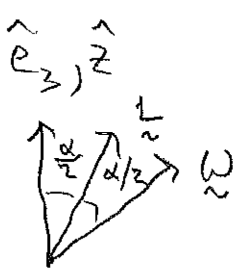
(g) State in words the meaning of $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$. $= \underset{\sim}{\Omega}_s$

$\underset{\sim}{\Omega}_{space}$ is the angular velocity at which the $\underset{\sim}{\omega}$ (and $\hat{e}_3$) vectors precess about the $\underset{\sim}{L}$ vector, as seen in the space frame.

$\underset{\sim}{\Omega}_{body}$ is the angular velocity at which the $\underset{\sim}{\omega}$ (and $\underset{\sim}{L}$) vectors precess about the $\hat{e}_3$ vector, as seen in the body frame.

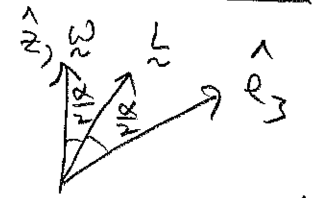
(h) In the $\alpha \ll 1$ limit (to first order in α, so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate. This should be some constant factor times α. A simple argument is sufficient here, no calculation.

at $t=0$



here $\measuredangle(\hat{z}, \hat{e}_3) = 0$

at $t = \frac{\pi}{\Omega_s}$



here $\measuredangle(\hat{z}, \hat{e}_3) \simeq \alpha$

In small α limit, tangent of $\measuredangle$ between $\hat{z}$ and $\underset{\sim}{L}$ is $\frac{\alpha}{2}$

$\sin\alpha / 2\cos\alpha \simeq \frac{\alpha}{2}$

(i) At what time t is this maximum deviation first reached?

$$t = \frac{\pi}{\Omega_s} = \frac{\pi}{\omega\sqrt{(2\cos\alpha)^2 + (\sin\alpha)^2}} \simeq \frac{\pi}{\omega\sqrt{4+\alpha^2}} \simeq \frac{\pi}{2\omega}$$

This time corresponds to $\frac{1}{2}$ period of Ω_{space}, which is approximately $\frac{1}{4}$ period of ω.

(j) Feynman's anecdote about a dinner plate tossed through the air in a Cornell cafeteria states, "when the angle [α] is very slight, the [red Cornell] medallion [on the plate] rotates twice as fast as the wobble rate." Was he remembering correctly? (Explain briefly.)

The rotation of the spinning medallion about the $\hat{e}_3$ axis is described by $\omega_3 = \omega\cos\alpha \simeq \omega$.

The "wobbling" (projected view of precession, seen from the side) of the plate axis $\hat{e}_3$ about the vertical axis $\hat{z}$ is described by $\Omega_{space} \simeq 2\omega$.

So the plate in fact wobbles 2x as fast as it spins. Feynman mis-remembered (40 years later) which way the factor of 2 had gone.

Problem 2. *Hamiltonian treatment of a spherical pendulum.*
The "spherical pendulum" is just a simple pendulum that is free to move in any sideways direction. (By contrast a "simple pendulum" — unqualified — is confined to a single vertical plane.) The bob of a spherical pendulum moves on a sphere, centered on the point of support with radius $r = R$, the length of the pendulum. A convenient choice of coordinates is spherical polars, r, θ, ϕ, with the origin at the point of support and the polar axis pointing straight down. The two variables θ and ϕ make a good choice of generalized coordinates.

(a) Write the Lagrangian in terms of θ, $\dot\theta$, ϕ, and $\dot\phi$.

$$U = -mgR\cos\theta$$

$$T = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}m(R\sin\theta)^2\dot\phi^2$$

$$\mathcal{L} = T - U = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}mR^2(\sin^2\theta)\dot\phi^2 + mgR\cos\theta$$

(b) Write the Hamiltonian in terms of θ, p_θ, ϕ, and p_ϕ. [In case it saves you some time, I can tell you that this choice of coordinates is indeed "natural."]

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = mR^2\dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{mR^2}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = mR^2(\sin^2\theta)\dot\phi \Rightarrow \dot\phi = \frac{p_\phi}{mR^2\sin^2\theta}$$

since coords θ, ϕ are "natural," $\mathcal{H} = T + U$

$$\mathcal{H} = \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta$$

(c) Which generalized coordinate is "ignorable?" Write both a name and an expression for the corresponding conserved quantity.

ϕ is ignorable, as $\frac{\partial \mathcal{L}}{\partial \phi} = 0$

$\Rightarrow \; P_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is a constant of the motion

P_ϕ is the z component of angular momentum.

$$P_\phi = m(R\sin\theta)^2 \dot{\phi}$$

(d) Write all four Hamilton's equations for this system.

$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mR^2\sin^2\theta} \quad ✓$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = \frac{P_\phi^2 \cos\theta}{mR^2\sin^3\theta} - mgR\sin\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mR^2} \quad ✓$$

These 2 equations simply reproduce the definitions of P_θ and P_ϕ. ✓

(e) Combine two of the four equations to write an expression for $\ddot{\theta}$.

$$\ddot{\theta} = \frac{d}{dt}\left(\dot{\theta}\right) = \frac{d}{dt}\left(\frac{P_\theta}{mR^2}\right) = \frac{\dot{P}_\theta}{mR^2}$$

$$\ddot{\theta} = \frac{P_\phi^2 \cos\theta}{(mR^2)^2\sin^3\theta} - \frac{g\sin\theta}{R}$$

(f) Suppose that we let $\theta = \theta_0 + \epsilon$, so that $\ddot{\epsilon} = \ddot{\theta} = f(\theta) = f(\theta_0 + \epsilon) \approx f(\theta_0) + \epsilon f'(\theta_0)$. What value would $f(\theta_0)$ have for circular orbits, in which θ would remain constant: $\theta \equiv \theta_0$? (Usually writing this condition lets you solve for θ_0, but don't spend time doing that here.)

$$f(\theta_0) = 0$$

(g) For small oscillations of θ with respect to θ_0, how would the frequency of small oscillations relate to $f'(\theta_0)$? Do you expect $f'(\theta_0)$ to be positive or negative? [There is nothing to solve here. We are just speaking in general terms.]

$$\left.\begin{aligned} \ddot{\varepsilon} &= \varepsilon f'(\theta_0) \\ \ddot{\varepsilon} &= -\Omega^2 \varepsilon \end{aligned}\right\} \Rightarrow \Omega = \sqrt{-f'(\theta_0)}$$

need $f'(\theta_0) < 0$

(h) In general (not directly related to the above problem), what must be true of the relationship between the inertial Cartesian coordinates (e.g. x, y) and the generalized coordinates (e.g. q_1, q_2) for the Hamiltonian $\mathcal{H}(q_1, q_2, p_1, p_2, t)$ to equal the total energy of the system?

$x = x(q_1, q_2)$ does not depend on $\dot{q}_1, \dot{q}_2, t$

Same for y. q_1, q_2 are called "natural" coordinates if they have this property.

(i) If $\partial \mathcal{L}/\partial t = 0$, what statement can be made about $\mathcal{H}$?

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

$\mathcal{H}$ is a constant of the motion if $\frac{\partial \mathcal{L}}{\partial t} = 0$.

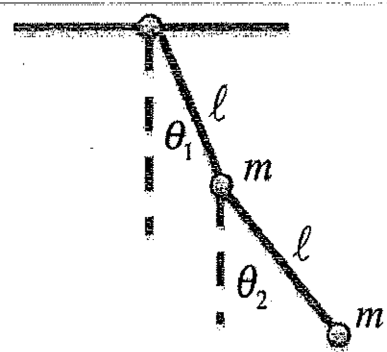
So $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \frac{d\mathcal{H}}{dt} = 0$

(total time derivative: stronger statement than partial time derivative)

Problem 3.

Consider the double pendulum consisting of two bobs confined to move in a plane. The rods are of equal length ℓ, and the bobs have equal mass m. The generalized coordinates used to describe the system are θ_1 and θ_2, the angles that the rods make with the vertical.

(a) Write the Lagrangian for the system. (This could be an opportunity to save some tedious algebra by writing $(\mathbf{v}_1 + \mathbf{v}_2)^2 = v_1^2 + v_2^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2$.)



$$U = -mg\ell\cos\theta_1 - m(g\ell\cos\theta_1 + g\ell\cos\theta_2)$$

$$U = -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2$$

$$T = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$v_1^2 = \ell^2\dot{\theta}_1^2 \qquad v_{2\,wrt\,1}^2 = \ell^2\dot{\theta}_2^2 \qquad \text{cosine}(\underset{\sim}{v_1}, \underset{\sim}{v_{2wrt1}}) = \cos(\theta_1 - \theta_2)$$

$$v_2^2 = (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))\ell^2$$

$$\mathcal{L} = \frac{1}{2}m\ell^2(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + mg\ell(2\cos\theta_1 + \cos\theta_2)$$

$$\cos\theta_1 \simeq 1 - \frac{1}{2}\theta_1^2$$

(b) Next, simplify your Lagrangian from part (a) by assuming that angles θ_1 and θ_2 are both small. Keep terms up to second order (total) in the angles, the angular velocities, and their products.

$$\mathcal{L} = \frac{1}{2}ml^2\left(2\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\right) - mgl\theta_1^2 - \frac{1}{2}mgl\theta_2^2$$

(c) Find the two Lagrange equations of motion, which will be a set of coupled, linear differential equations. [You don't need to solve these equations!]

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -2mgl\theta_1 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_1}\right) = \frac{d}{dt}\left(\frac{1}{2}ml^2(4\dot\theta_1 + 2\dot\theta_2)\right)$$

$$-2mgl\theta_1 = 2ml^2\ddot\theta_1 + ml^2\ddot\theta_2$$

$$\boxed{2\ddot\theta_1 + \ddot\theta_2 = -\frac{2g}{l}\theta_1}$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = -mgl\theta_2 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_2}\right) = \frac{d}{dt}\left(\frac{1}{2}ml^2(2\dot\theta_2 + 2\dot\theta_1)\right)$$

$$= ml^2(\ddot\theta_2 + \ddot\theta_1)$$

$$\boxed{\ddot\theta_1 + \ddot\theta_2 = -\frac{g}{l}\theta_2}$$

Problem 4.

We're going to work out the equations of motion for a Foucault pendulum. First we'll use the Newtonian method, then we'll use the Lagrangian method to find the same result. Since this is not an adaptation of a homework problem, I will try to guide you through it.

A Foucault pendulum consists of a very heavy mass m (the "bob") suspended from a very tall ceiling by a light wire of length L. This arrangement allows the pendulum to swing freely for a very long time and to move in both the east-west and the north-south directions. Let $\hat{x}$ point east, let $\hat{y}$ point north, and let $\hat{z}$ point up (away from Earth's center). Let $(x, y, z) = (0, 0, 0)$ be the equilibrium position of the bob. The very tall ceiling allows us to approximate $x \ll L$, $y \ll L$, $z \approx 0$.

(a) If we ignore Earth's rotation, we find $\ddot{x}$ and $\ddot{y}$ equations of motion that are completely uncoupled. To save you the hassle of working it out, the restoring force in x (the x component of the wire tension acting on the bob) has magnitude mgx/L, and the restoring force in y has magnitude mgy/L. Write down the two (uncoupled) equations of motion for $\ddot{x}$ and $\ddot{y}$, each of which should independently describe the familiar motion of an ordinary plane pendulum at small amplitude.

$$\ddot{x} = -\frac{g}{L}x$$

$$\ddot{y} = -\frac{g}{L}y$$

(b) Our favorite velocity-dependent fictitious force will couple these two equations of motion, adding one term to the expression for $\ddot{x}$ and adding one term to the expression for $\ddot{y}$. The added terms will depend on some combination of the velocities $\dot{x}$ and $\dot{y}$, the colatitude θ, and Earth's angular velocity of rotation Ω. Using your knowledge of fictitious forces, find the expressions for $\ddot{x}$ and $\ddot{y}$.

$$\ddot{x} = -\frac{g}{L}x + 2(\underset{\sim}{v} \times \underset{\sim}{\Omega})_x = -\frac{g}{L}x + 2(\dot{y}\Omega\cos\theta)$$

$$\ddot{y} = -\frac{g}{L}y + 2(\underset{\sim}{v} \times \underset{\sim}{\Omega})_y = -\frac{g}{L}y + 2(-\dot{x}\Omega\cos\theta)$$

$$\ddot{x} = -\omega_0^2 x + 2\dot{y}\Omega_z$$

$$\ddot{y} = -\omega_0^2 y - 2\dot{x}\Omega_z$$

where $\Omega_z = \Omega\cos\theta$

$\omega_0^2 = \frac{g}{L}$

(c) Let's suppose that at $t = 0$ we displace the bob a distance $A \ll L$ due east of its equilibrium position and release it from rest. Check that your equations of motion are satisfied by

$$x(t) = A\cos(\Omega_z t)\cos(\omega_0 t) \qquad y(t) = -A\sin(\Omega_z t)\cos(\omega_0 t)$$

where $\Omega_z = \Omega\cos\theta$ and $\omega_0 = \sqrt{g/L}$. Since $\Omega \ll \omega$, you should **ignore** any terms of order Ω^2.

$$\dot{x} = -A\Omega_z \sin(\Omega_z t)\cos(\omega_0 t) - A\omega_0 \cos(\Omega_z t)\sin(\omega_0 t)$$

$$\ddot{x} = A\omega_0\Omega_z \sin(\Omega_z t)\sin(\omega_0 t) + A\omega_0\Omega_z \sin(\Omega_z t)\sin(\omega_0 t) - A\omega_0^2 \cos(\Omega_z t)\cos(\omega_0 t)$$

$$\dot{y} = -A\Omega_z \cos(\Omega_z t)\cos(\omega_0 t) + A\omega_0 \sin(\Omega_z t)\sin(\omega_0 t)$$

$$\ddot{y} = A\omega_0\Omega_z \cos(\Omega_z t)\sin(\omega_0 t) + A\omega_0\Omega_z\cos(\Omega_z t)\sin(\omega_0 t) + A\omega_0^2 \sin(\Omega_z t)\cos(\omega_0 t)$$

$$\ddot{x} + \omega_0^2 x - 2\dot{y}\Omega_z = A\omega_0\Omega_z \cancel{\sin(\Omega_z t)}\sin(\omega_0 t) + A\omega_0\Omega_z \cancel{\sin(\Omega_z t)}\sin(\omega_0 t) - A\omega_0^2 \cancel{\cos(\Omega_z t)}\cos(\omega_0 t) + \omega_0^2 A\cancel{\cos(\Omega_z t)}\cos(\omega_0 t) - 2\Omega_z\left(A\omega_0 \cancel{\sin(\Omega_z t)}\sin(\omega_0 t)\right) = 0 \quad \checkmark$$

$$\ddot{y} + \omega_0^2 y + 2\dot{x}\Omega_z = +A\omega_0\Omega_z \cos(\Omega_z t)\sin(\omega_0 t) + A\omega_0\Omega_z\cos(\Omega_z t)\sin(\omega_0 t) + A\omega_0^2\sin(\Omega_z t)\cos(\omega_0 t) - \omega_0^2 A\sin(\Omega_z t)\cos(\omega_0 t) + 2\Omega_z\left(-A\Omega_z \sin(\Omega_z t)\cos(\omega_0 t) - A\omega_0\cos(\Omega_z t)\sin(\omega_0 t)\right) = 0 \quad \checkmark$$

(d) Now use the Lagrangian approach to find the same $\ddot{x}$ and $\ddot{y}$ equations of motion that you found in part (b), in terms of the same x (east) and y (north) coordinates. Because the pendulum's amplitude $A \ll L$, you can approximate the potential energy as $U = mg(x^2 + y^2)/(2L)$. (If you had time to work it out yourself, the factor of 2 would come from the cosine of the small angle $\sqrt{x^2 + y^2}/L$.) To write the kinetic energy in the inertial "space" frame, you will need the vector $\boldsymbol{r} = (x, y, R)$ that points from Earth's center to the bob's position, where R is Earth's radius. Write $\mathcal{L}$, then find $\ddot{x}$ and $\ddot{y}$, which should agree with part (b). **Ignore** (as early as you like) any terms of order Ω^2.

$$V_x = \dot{x} + (\underset{\sim}{\Omega} \times (x, y, R))_x = \dot{x} + \Omega_y R - \Omega_z y$$

$$V_y = \dot{y} + (\underset{\sim}{\Omega} \times (x, y, R))_y = \dot{y} + \Omega_z x - 0$$

$$V_x^2 = \dot{x}^2 + 2\dot{x}\Omega_y R - 2\dot{x}\Omega_z y - \mathcal{O}(\Omega^2)$$

$$V_y^2 = \dot{y}^2 + 2\dot{y}\Omega_z x + \mathcal{O}(\Omega^2)$$

$$U = \frac{mg}{2L}(x^2 + y^2)$$

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\Omega_y R - 2\dot{x}\Omega_z y + 2\dot{y}\Omega_z x\right) - \frac{mg}{2L}(x^2 + y^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = m\dot{y}\Omega_z - \frac{mg}{L}x = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x} + m\Omega_y R - m\Omega_z y\right) = m\ddot{x} - m\Omega_z \dot{y}$$

$$\Rightarrow \ddot{x} = 2\dot{y}\Omega_z - \frac{g}{L}x \quad \checkmark$$

$$\frac{\partial \mathcal{L}}{\partial y} = -m\dot{x}\Omega_z - \frac{mgy}{L} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{d}{dt}\left(m\dot{y} + m\Omega_z x\right) = m\ddot{y} + m\Omega_z \dot{x}$$

$$\Rightarrow \ddot{y} = -2\dot{x}\Omega_z - \frac{g}{L}y \quad \checkmark$$

Congratulations! You have solved the quintessential fictitious-force problem!

Possibly useful equations.

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega}\times\boldsymbol{Q}$$

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}}\times\boldsymbol{\Omega} + m\left(\boldsymbol{\Omega}\times\boldsymbol{r}\right)\times\boldsymbol{\Omega} = \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m\Omega^2\rho\,\hat{\boldsymbol{\rho}}$$

For a uniform solid cylinder of radius R about its symmetry axis, $I = mR^2/2$. For a uniform thin rod of length L about its center (perpendicular to the rod axis), $I = mL^2/12$. For a rectangular plate about its center (rotation axis normal to plate), $I = m(a^2+b^2)/12$, where a and b are the short and long side lengths.

For a free symmetric top, $\boldsymbol{\Omega}_s = \boldsymbol{L}/\lambda_1$. One way to prove this is to notice that $\frac{\boldsymbol{L}}{\lambda_1} = \boldsymbol{\omega} + \frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3\hat{\boldsymbol{e}}_3$, then to see how $\hat{\boldsymbol{e}}_3$ evolves in time.

Euler equations:

$$\begin{aligned}\tau_1 &= \lambda_1\dot{\omega_1} - (\lambda_2-\lambda_3)\omega_2\omega_3\\ \tau_2 &= \lambda_2\dot{\omega_2} - (\lambda_3-\lambda_1)\omega_3\omega_1\\ \tau_3 &= \lambda_3\dot{\omega_3} - (\lambda_1-\lambda_2)\omega_1\omega_2\end{aligned}$$

If $\boldsymbol{\tau} = 0$ and $\lambda_1 = \lambda_2$ then the Euler equations reduce to the simpler form

$$\dot{\omega_3} = \frac{\lambda_1-\lambda_1}{\lambda_3}\,\omega_1\omega_2 = 0$$

$$\begin{aligned}\dot{\omega_1} &= \frac{\lambda_1-\lambda_3}{\lambda_1}\,\omega_2\omega_3 = -\left(\frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3\right)\omega_2 = -\,\Omega_b\,\omega_2\\ \dot{\omega_2} &= \frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3\omega_1 = \left(\frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3\right)\omega_1 = \Omega_b\,\omega_1\end{aligned}$$

so we can represent the precession of the $\boldsymbol{\omega}$ vector as an angular velocity vector $\boldsymbol{\Omega}_b$ with

$$\boldsymbol{\Omega}_b = \left(\frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3\right)\,\hat{\boldsymbol{e}}_3.$$

You shouldn't need to use this, but here it is anyway, to remind you of the definitions of θ and ϕ:

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$