Physics 351, Spring 2018, Homework #1.

Due at start of class, Friday, January 26, 2018

Schedule and handouts are at `positron.hep.upenn.edu/p351`

Please write your name only on the $\boxed{\text{VERY LAST PAGE}}$ of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

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1. The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. (a) Use Mathematica (or other software if you prefer) to solve the differential equation for the case that the board is released from $\phi_0 = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of $\phi(t)$ for two or three periods. (b) On the same picture, plot the approximate solution (1.57) with the same $\phi_0 = 20°$. Compare your two graphs. (c) Repeat parts (a) and (b) using the initial value $\phi_0 = \pi/2$ and compare. You will need to learn to use Mathematica's `NDSolveValue` command and to plot the solution that it provides using the `Plot` command. The `Plot` command can also graph the approximate solution (1.57). The graph is most informative if you overlay the numerical solution and the approximate solution on the same axes for direct comparison. I'll illustrate in class how to do these things in Mathematica.

2. There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in Chapter 2) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. (a) Write down Newton's second law and separate the variables by rewriting it as $m\frac{\mathrm{d}v}{F(v)} = \mathrm{d}t$. Now integrate both sides of this equation and show that

$$t = m\int_{v_0}^{v} \frac{\mathrm{d}v'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. (b) Use this method to solve the special case that $F(v) = F_0$, a constant force, and notice that you basically get a "freshman physics" result. (c) Next, use the same method to solve for the case in which a mass m has velocity v_0 at time $t = 0$ and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Find v in terms of the time t and the other given parameters. At what time (if any) will the mass come to rest?

3. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\mathrm{d}x'.$$

Hint: use the chain rule to prove the following handy relation: If you regard v as a function of x, then

$$\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}x}.$$

Use the above relation to rewrite Newton's second law in the separated form $m\ \mathrm{d}(v^2) = 2F(x)\ \mathrm{d}x$ and then integrate from x_0 to x. Notice that the result may look familiar ("freshman physics") in the case that $F(x)$ is actually a constant. (You should recognize your solution as a statement about kinetic energy and work.)

4. Use the method of Problem 2 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_0 e^{v/V}$, where F_0 and V are constants. (a) Find $v(t)$ if the initial velocity is $v_0 > 0$ at time $t = 0$. (b) At what time does the mass come instantaneously to rest? (c) By integrating $v(t)$, you can find $x(t)$. Do this and show that the distance the mass travels before coming instantaneously to rest is

$$\Delta x = \frac{mV^2}{F_0}\left(1 - \nu e^{-\nu} - e^{-\nu}\right)$$

where $\nu \equiv v_0/V$. (Feel free to do the integral with Mathematica.)

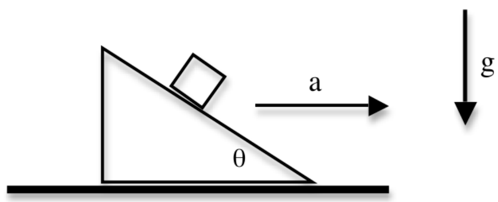
5. A basketball has mass $m = 600$ g and diameter $D = 24$ cm. (a) What is its terminal speed in air? (b) If it is dropped from a 30 m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

6. Two elephants, each of mass m, are standing at one end of a stationary railroad flatcar of mass M, which has frictionless wheels. Either elephant can run to the other end of the flatcar and jump off with the same speed u

(relative to the car). (a) Use conservation of momentum to find the speed of the recoiling car if the two elephants run and jump simultaneously. (b) What is it if the second elephant starts running only after the first has already jumped? Which procedure gives the greater speed to the car? Hint: The speed u is the speed of either elephant, *relative to the car*, just after it has jumped; it has the same value for either elephant and is the same in parts (a) and (b).

7. A block rests on a wedge whose incline has coefficient of static friction μ and is at angle θ from the horizontal. (See figure below.) (a) Assuming that the wedge is fixed in position, find the maximum value of θ such that the block remains motionless on the wedge. (b) Now suppose that $\tan\theta > \mu$, so that the block slides downhill if the wedge is motionless. Also suppose that **the wedge** is accelerating to the right with constant acceleration a. Find the minimum and maximum values of a for which the block can remain motionless w.r.t. the wedge.

Figure for problem 7:



8. Consider a small frictionless puck perched at the top of a fixed sphere of radius R. If the puck is given a tiny nudge so that it begins to slide down, through what vertical height will it descend before it leaves the surface of the sphere? [Hint: At what value of the normal force between sphere and puck does the puck leave the sphere?]

9. Use spherical polar coordinates r, θ, ϕ to find the CM of a uniform solid hemisphere of radius R, whose flat face lies in the xy plane with its center at the origin. To do this, you need to remember (or convince yourself) that the element of volume in spherical polars is $\mathrm{d}V = r^2\ \mathrm{d}r \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$.

10. A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. (a) Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle. What is the particle's angular velocity now? (b) Now let's see what happens during the pull described in part (a). Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0. Starting at $t = 0$, I pull the string with constant velocity v so that the radial distance (r) to the mass decreases. Draw a force diagram for the mass and find a differential equation for $\omega(t)$. Find $\omega(t)$ and also find the force $F(t)$ that I need to exert on the string. [Hint: one component of the force exerted on m by the string is always zero.]

11. Near to the point where I am standing on the surface of Planet X, the gravitational force on a mass m is vertically down but has magnitude $m\gamma y^2$ where γ is a constant and y is the mass's height above the horizontal ground. (a) Find the work done by gravity on a mass m moving from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, and use your answer to show that gravity on Planet X, although most unusual, is still conservative. Find the corresponding potential energy. (b) Still on the same planet, I thread a bead on a curved, frictionless, rigid wire, which extends from ground level to a height h above the ground. Show clearly in a picture the forces on the bead when it is somewhere on the wire. (Just name the forces so it's clear what they are; don't worry about their magnitude.) Which of the forces are conservative and which are not? (c) If I release the bead from rest at a height h, how fast will it be going when it reaches the ground?

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(extra credit problems below — some hard, some quite easy!)

XC1. Optional/extra-credit. Suppose that the basketball of Problem 5 is thrown from a height of 3 m with initial velocity $\mathbf{v}_0 = 18$ m/s at 45° above the horizontal. (a) Use Mathematica (or some other system that you already know) to solve the equations of motion (2.61) for the ball's position (x, y) and plot the trajectory. Also plot the corresponding trajectory in the absence of air resistance. (b) Use your plot to find how far the ball travels in the horizontal direction before it hits the floor. Compare with the corresponding range in a vacuum.

XC2. Optional/extra-credit. The equation (2.39) for the range of a projectile in a linear medium cannot be solved analytically in terms of elementary functions. If you put in numbers for the several parameters, then it *can* be solved numerically using Mathematica (or similar). To practice this, do the following: Consider a projectile launched at angle θ above the horizontal ground with initial speed v_0 in a linear medium. Choose units such that $v_0 = 1$ and $g = 1$. Suppose also that the terminal speed $v_{\text{ter}} = 1$. (With $v_0 = v_{\text{ter}}$, air resistance should be fairly important.) We know that in a vacuum, the maximum range occurs at $\theta = \pi/4 \approx 0.75$. (a) What is the maximum range in a vacuum? (b) Now solve (2.39) for the range in the given medium at the same angle $\theta = 0.75$. (c) Once you have your calculation working, repeat it for some selection of values of θ within which the maximum range probably lies — e.g. you could try $\theta = 0.4, 0.5, \cdots, 0.8$. (d) Based on these results, choose a smaller interval for θ where you're sure the maximum lies and repeat the process. Repeat it again if necessary until you know the maximum range and the corresponding angle to two significant figures. Compare with the vacuum values.

Mathematica hints: I started this by typing in Equation (2.39) as it appears in the book and giving this equation the name "eq1". Since Mathematica's built-in functions and variables begin with capital letters, all of my own variables start with lowercase letters.

```
eq1 = (vy0 + vter)*r/vx0 + vter*tau*Log[1-r/(vx0*tau)]==0
```

Then I defined "eq2" to be the same equation with a few handy replacements, using Mathematica's `ReplaceAll` operator, whose shorthand is `/.` (slash dot), which when I read it sounds like "such that."

```
eq2 = eq1 /. {tau->vter/g, vx0->v0*Cos[th], vy0->v0*Sin[th]}
```

Then I defined "eq3" to be eq2 with a few more replacements:

```
eq3 = eq2 /. {v0->1, vter->1, g->1}
```

which Mathematica then writes as

```
Log[1 - r*Sec[th]] + r*Sec[th]*(1 + Sin[th]) == 0
```

To solve this for $\theta = 0.75$, I do one more replacement and use the `Solve` function: `Solve[eq3 /. th->0.75]` which finds $r = 0.499597$, which you can check by plugging in numbers. (I take only the left-hand side of “eq3” by taking the “`First`” element of the equation.)

```
First[eq3] /. {th->0.75, r->0.4996}
```

You can repeat the `Solve` step for other values of θ. You might also want to check that using `vter->1000` gives you approximately the range you calculated (in these funny units) for part (a). By the way, if you are already a Mathematica expert and you know a more straightforward (but still understandable by a beginner) way of solving this problem, please send it to me!

XC3. Optional/extra-credit. A ball is thrown with initial speed v_0 up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. (a) Write down Newton's second law using these axes and find the ball's position as a function of time. (b) Show that the ball lands a distance R from its launch point, where $R = 2v_0^2 \sin\theta \cos(\theta + \phi)/(g \cos^2 \phi)$. (c) Show that for a given v_0 and ϕ, the maximum possible range up the inclined plane is $R_{\max} = v_0^2/[g(1 + \sin\phi)]$. (d) For level ground, it is well known that the maximum range occurs for a projectile thrown at 45°. Can you give a simple statement of what angle corresponds to the maximum range for the projectile on an incline?

XC4. Optional/extra-credit. A cannon shoots a ball at an angle θ above the horizontal ground. (a) Neglecting air resistance, use Newton's second law to find the ball's position as a function of time. (Use axes with x measured horizontally and y measured vertically.) (b) Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight? [Hint: Using your solution to part (a), you can write down $r^2 = x^2 + y^2$, and then find the condition that r^2 is always increasing.]

XC5. Let $\mathbf{u}$ be an arbitrary fixed unit vector. Show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2.$$

Notice that the first term picks out the part of $\mathbf{b}$ that is parallel to $\mathbf{u}$, while the second term picks out the part of $\mathbf{b}$ that is perpendicular to $\mathbf{u}$.

XC6. If $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}).$$

Hint: Note that the derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$ distributes over vector products (dot product, cross product) analogously to the way it does over ordinary products. So for example,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\cdot\mathbf{b}) = \dot{\mathbf{a}}\cdot\mathbf{b} + \mathbf{a}\cdot\dot{\mathbf{b}}.$$

XC7. The two vectors **a** and **b** lie in the xy plane and make angles α and β with the x axis. (a) By evaluating the dot product $\mathbf{a}\cdot\mathbf{b}$ in two ways [namely using equations (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a}\times\mathbf{b}$ prove that

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

XC8. Problems XC8 and XC9 are (embarrassingly easy) problems from Chapter 2 that I think are worth doing because they walk you through things that are worth knowing about air resistance. The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. (a) Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ρ, show that the rate at which the projectile encounters fluid (mass/time) is ρAv. (b) Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is ρAv^2. (c) More realistically, as it turns out, the force takes the form $\boxed{f_{\text{quad}} = \kappa\rho Av^2}$ where $\kappa < 1$ depends on the shape of the projectile. Show that the boxed equation reproduces $f_{\text{quad}} = cv^2 = \gamma D^2v^2$, where the density of air at STP is $\rho = 1.29\ \text{kg/m}^3$ and given that $\kappa = 1/4$ for a sphere. Check that you reproduce the textbook's value $\gamma = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4$.

XC9. (a) The origin of the linear drag force on a sphere in a fluid is the viscosity of the fluid. According to Stokes's law, the viscous drag on a sphere is $f_{\text{lin}} = 3\pi\eta Dv$ where η is the viscosity[1] of the fluid, D is the sphere's diameter, and v its speed. Given the viscosity of air at STP, $\eta = 1.7\times10^{-5}\ \text{N}\cdot\text{s/m}^2$, show that this expression reproduces $f_{\text{lin}} = bv = \beta Dv$, where $\beta = 1.6\times10^{-4}\ \text{N}\cdot\text{s/m}^2$. (b) The quadratic drag force on a moving sphere in a fluid is given by the boxed equation in Problem XC8. Show that the ratio of drag forces can be written as

[1]To define viscosity η, imagine a wide channel along which fluid is flowing (x direction) such that the velocity v is zero at the bottom ($y = 0$) and increases toward the top ($y = h$), so that successive layers of fluid slide across one another with a velocity gradient $\mathrm{d}v/\mathrm{d}y$. The force F with which an area A of any one layer drags the fluid above it is proportional to A and to $\mathrm{d}v/\mathrm{d}y$, and η is defined as the constant of proportionality: $F = \eta A\mathrm{d}v/\mathrm{d}y$.

$f_{\text{quad}}/f_{\text{lin}} = R/48$, where the dimensionless *Reynolds number*[2] is $R = Dv\rho/\eta$, where ρ is the fluid's density. Clearly the Reynolds number is a measure of the relative importance of the two kinds of drag.

XC10. Prove that the magnetic forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ between two steady current loops (for which there is no electromagnetic wave to carry away momentum) obey Newton's third law. Hints: Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $\mathrm{d}\mathbf{r}_1$ and $\mathrm{d}\mathbf{r}_2$ are short segments of the loops, then according to the Biot-Savart law, the force on $\mathrm{d}\mathbf{r}_1$ due to $\mathrm{d}\mathbf{r}_2$ is

$$\frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2}\,\mathrm{d}\mathbf{r}_1 \times (\mathrm{d}\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$, and $\hat{\mathbf{s}} = \mathbf{s}/s$. The force $\mathbf{F}_{12}$ is found by integrating around both loops. Start by writing down the force on $\mathrm{d}\mathbf{r}_1$ due to $\mathrm{d}\mathbf{r}_2$, and expand it using the "BAC-CAB" rule. Do the same thing for the force on $\mathrm{d}\mathbf{r}_2$ due to $\mathrm{d}\mathbf{r}_1$. Each force will have two terms. One term in each force will involve $\mathrm{d}\mathbf{r}_1 \cdot \mathrm{d}\mathbf{r}_2$, and you can show that they are the negative of each other. You should be able to show that the other term in each force is of the form $\oint \nabla f \cdot \mathrm{d}\mathbf{r} = \oint \mathrm{d}f = 0$, i.e. the line integral of the gradient of a scalar function is zero around a closed path. This argument thus establishes that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Some hints: Let $\mathbf{s} \equiv \mathbf{r}_1 - \mathbf{r}_2$. Using $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, we can write $\mathrm{d}\mathbf{r}_1 \times (\mathrm{d}\mathbf{r}_2 \times \frac{\hat{\mathbf{s}}}{s^2}) = \mathrm{d}\mathbf{r}_2(\mathrm{d}\mathbf{r}_1 \cdot \frac{\hat{\mathbf{s}}}{s^2}) - \frac{\hat{\mathbf{s}}}{s^2}(\mathrm{d}\mathbf{r}_1 \cdot \mathrm{d}\mathbf{r}_2)$. Also notice that $\nabla(\frac{1}{r}) = -\frac{\hat{\mathbf{r}}}{r^2}$, and so $\nabla_1(\frac{1}{s}) = -\frac{\hat{\mathbf{s}}}{s^2} = -\frac{\mathbf{s}}{s^3} = -\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}$. One way to prove that is $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{(2x,2y,2z)}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2}$. Also note that $\oint \mathrm{d}\mathbf{r}_1 \cdot \nabla_1(\frac{1}{s}) = 0$ because $\nabla \times (\nabla f) = 0$ for any scalar function f. In general there will not be very much tricky vector calculus of this sort in Physics 351.

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[2] The factor 1/48 is for a sphere.