Physics 351, Spring 2018, Homework #2.

Due at start of class, Friday, February 2, 2018

Please write your name only on the VERY LAST PAGE of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

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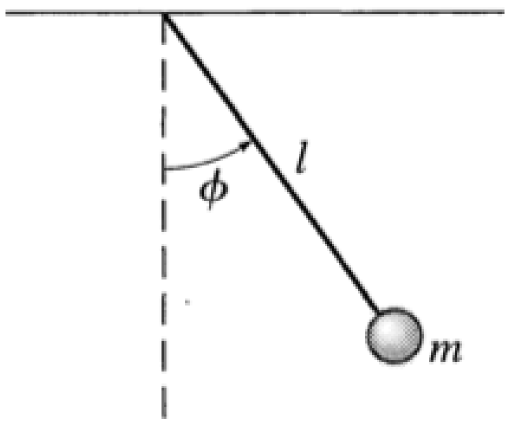
1. Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at angle γ to the horizontal. The instantaneous point of contact between the disk and the incline is called P. (a) Draw a free-body diagram, showing all forces on the disk. (b) Find the linear acceleration $\dot{v}$ of the disk by applying the result $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ for rotation about P. (Remember to use the parallel-axis theorem for rotation about a point on the circumference.) (c) Derive the same result by applying $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ to the rotation about the CM. (In this case there will be an extra unknown, the force of friction, which you can eliminate using the equation of motion of the CM.)

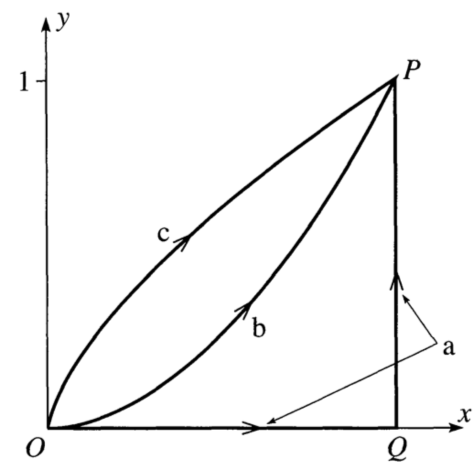
2. A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U = kr^n$. (a) Prove the **virial theorem**, that $T = nU/2$. (b) What does the virial theorem (assuming that it generalizes beyond circular orbits) imply for $n = -1$ (the gravitational Kepler problem) and for $n = 2$ (the harmonic-oscillator problem)? [The motivation for including this problem is that you may at some point see the virial theorem invoked, e.g. in a quantum mechanics course, to argue that $\langle T \rangle = -\frac{1}{2} \langle U \rangle$ for the Kepler problem (or the analogous hydrogen atom problem) and that $\langle T \rangle = \langle U \rangle$ for the simple harmonic oscillator.]

3. A chain of mass M and length L is suspended vertically with its lowest end just barely touching a scale. The chain is released and falls onto the scale. What is the reading on the scale when a length x of the chain has fallen? [Hint: The reading on the scale equals the normal force exerted by the scale on the chain. Consider the motion of the center-of-mass of the chain. The maximum reading is $3Mg$.]

4. Which of the following forces are conservative? (a) $\boldsymbol{F} = k\ (x, 2y, 3z)$ where k is a constant. (b) $\boldsymbol{F} = k\ (y, x, 0)$. (c) $\boldsymbol{F} = k\ (-y, x, 0)$. For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\boldsymbol{F} = -\nabla U$.

5. An interesting one-dimensional system is the simple pendulum, consisting of a point mass m fixed to the end of a massless rod (length l), as shown in the left figure below. The pendulum's position can be specified by its angle ϕ from the equilibrium position. (a) Prove that the pendulum's potential energy is $U(\phi) = mgl(1 - \cos\phi)$. Then write down the total energy E as a function of ϕ and $\dot{\phi}$. (b) Show that requiring the total energy E to be independent of time ($\mathrm{d}E/\mathrm{d}t = 0$) gives the equation of motion for ϕ, and that this EOM is just the familiar $\Gamma = I\alpha$, where Γ is torque, I is moment of inertia, and $\alpha = \ddot{\phi}$. (c) Assuming that $\phi(t) \ll 1$, solve for $\phi(t)$ and show that the motion is periodic with period $\tau_0 = 2\pi\sqrt{l/g}$.





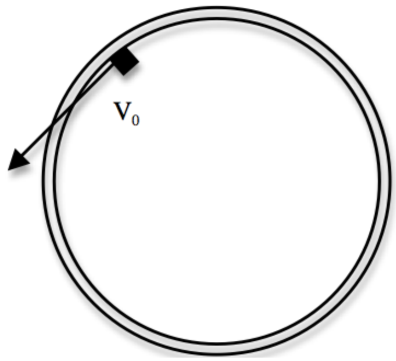
6. Evaluate the work done

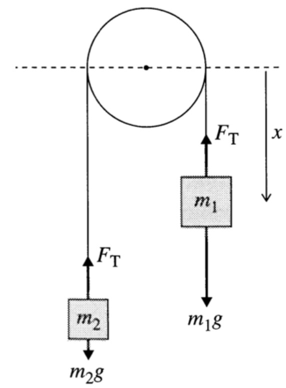
$$W = \int_O^P \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_O^P (F_x \; \mathrm{d}x + F_y \; \mathrm{d}y)$$

by the two-dimensional force $\boldsymbol{F} = (x^2, 2xy)$ along the three paths joining the origin to the point $P = (1, 1)$ as shown in the above-right figure and defined as follows: (a) This path goes along the x axis to $Q = (1, 0)$ and then straight up to P. (b) On this path $y = x^2$, and you can write $\mathrm{d}y = 2x \; \mathrm{d}x$. (c) This path is given parameterically as $x = t^3$, $y = t^2$. In this case, convert the integral into an integral over t.

7. Consider a mass m on the end of a spring of Hooke's-law constant k and constrained to move along the horizontal x axis. If we place the origin at the spring's equilibrium position, the potential energy is $\frac{1}{2}kx^2$. At time $t = 0$ the mass is sitting at the origin and is given a sudden kick to the right so that it moves out to a maximum displacement at $x_{\max} = A$ and then continues to oscillate about the origin. (a) Write down the equation for conservation of energy and solve it to give the mass's velocity $\dot{x}$ in terms of the position x and the total energy E. (b) Show that $E = \frac{1}{2}kA^2$, and use this to eliminate E from your expression for $\dot{x}$. Use the result (4.58), $t = \int \mathrm{d}x'/\dot{x}(x')$, to find the time for the mass to move from the origin out to a position x. (c) Solve the result of part (b) to give x as a function of t and show that the mass executes simple harmonic motion with period $2\pi\sqrt{m/k}$.

8. A block of mass m slides on a frictionless (horizontal) table and is constrained to move along the inside of a ring of radius R, which is fixed to the table. At $t = 0$ the mass is moving (tangentially) along the inside of the ring with velocity v_0. The coefficient of kinetic friction between the block and ring is μ. Find the velocity $\dot{s}$ and position s (the arc length traveled) of the block as a function of time.





9. Consider the Atwood machine shown in the right figure above, where the pulley has radius R and moment of inertia I. (a) Write down the total energy of the two masses and the pulley in terms of the coordinate x and $\dot{x}$. (Remember the K.E. of the spinning wheel.) (b) Show (as is true for any conservative one-dimensional system) that you can obtain the EOM for x by differentiating the equation $E = $ const. Check that the EOM is the same as you would obtain by applying Newton's second law separately to the two masses and the pulley, and then eliminating the two unknown tensions from the three resulting equations. (This problem seems to be hinting toward the notion that writing down expressions for energies can lead us straightforwardly to the equations of motion — as we'll see in the Lagrangian formulation.)

10. The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

for $0 < r < \infty$, with U_0, R, and λ all positive constants. Find the equilibrium position r_0. Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const.} + \frac{1}{2}kx^2$. What is the natural angular frequency ω_0 for small oscillations?

11. Another interpretation of the Q of a resonance comes from the following: Consider the motion of a driven damped oscillator after any transients have died out, and suppose that it is being driven close to resonance, so that you can set $\omega = \omega_0$. (a) Show that the oscillator's total energy $(T + U)$ is $E = \frac{1}{2}m\omega^2 A^2$. (b) Show that the energy ΔE_{dis} dissipated during one cycle by the damping force F_{damp} is $2\pi m\beta\omega A^2$. (Remember power is Fv.) (c) Hence show that $Q \equiv \frac{\omega_0}{2\beta}$ is 2π times the ratio $E/\Delta E_{\text{dis}}$.

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(extra credit below)

XC1. Optional/extra-credit. A grenade is thrown with initial velocity $\boldsymbol{v}_0$ from the origin at the top of a high cliff, subject to negligible air resistance. (a) Using Mathematica (or your favorite alternative), plot the orbit, with the following parameters: $\boldsymbol{v}_0 = (4, 4)$, $g = 1$, and $0 \leq t \leq 4$ (and with x measured horizontally and y vertically up). Add to your plot suitable marks (dots or crosses, for example) to show the positions of the grenade at $t = 1, 2, 3, 4$. (b) At $t = 4$, when the grenade's velocity is $\boldsymbol{v}$, it explodes into two equal pieces, one of which moves off with velocity $\boldsymbol{v} + \Delta\boldsymbol{v}$. What is the velocity of the other piece? (c) Assuming that $\Delta\boldsymbol{v} = (1, 3)$, add to your original plot the paths of the two pieces for $4 \leq t \leq 9$. Insert marks to show their positions at $t = 5, 6, 7, 8, 9$. Find some way to show clearly that the CM of the two pieces continues to follow the original parabolic path.

XC2. Optional/extra-credit. A system consists of N masses m_α at positions $\boldsymbol{r}_\alpha$ relative to a fixed origin O. Let $\boldsymbol{r}'_\alpha$ denote the position of m_α relative to the CM; that is, $\boldsymbol{r}'_\alpha = \boldsymbol{r}_\alpha - \boldsymbol{R}$. (a) Make a sketch to illustrate this last equation. (b) Prove the useful relation that $\sum m_\alpha \boldsymbol{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious? (c) Use this relation to prove the result (3.28) that the rate of change of the angular momentum *about the CM* is equal to the total external torque about the CM. (This result is surprising since the CM may be accelerating, so that it is not necessarily a fixed point in any inertial frame.)

XC3. Optional/extra-credit. [Computer] A mass m confined to the x axis has potential energy $U = kx^4$ with $k > 0$. (a) Sketch this potential energy and qualitatively describe the motion if the mass is initially stationary at $x = 0$ and is given a sharp kick to the right at $t = 0$. (b) Use (4.58) to find the time for the mass to reach its maximum displacement $x_{\max} = A$. Give your answer as an integral over x in terms of m, A, and k. Hence find the period τ of oscillations of amplitude A as an integral. (c) By making a suitable change of variables in the integral, show that the period τ is inversely proportional to the amplitude A. (d) The integral of part (b) cannot be evaluated in terms of elementary functions, but it can be done numerically. Find the period for the case that $m = k = A = 1$.

XC4. Optional/extra-credit. (a) Verify the expression (Eq. 4.59) for the potential energy of the cube balanced on a cylinder in Example 4.7 (page 130). [Hint: To understand the $r\theta$ factor, imagine the cylinder rolling on the cube.] (b) Make graphs of $U(\theta)$ for $b = 0.9r$ and $b = 1.1r$, preferably by computer. (For simplicity, choose units such that r, m, and g all equal 1.) (c) Use your graphs to confirm the findings of Example 4.7 concerning the stability of the equilibrium at $\theta = 0$. Are there any other equilibrium points, and are they stable?

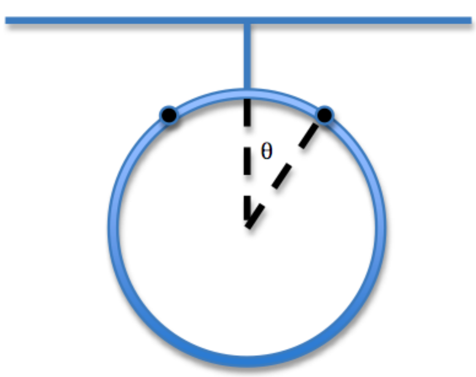
XC5. Optional/extra-credit. [Computer] Consider the simple pendulum of Problem 5. You can get an expression for the pendulum's period (good for both large and small oscillations) using the method of (Eq. 4.57), as follows: (a) Using $U(\phi) = mgl(1 - \cos\phi)$, find $\dot{\phi}$ as a function of ϕ. Next use $t = \int \mathrm{d}\phi/\dot{\phi}$ to write the time for the pendulum to travel from $\phi = 0$ to its maximum value Φ, and use this to show that the period of oscillation is

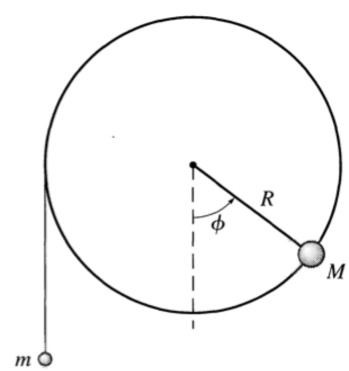
$$\tau = \frac{\tau_0}{\pi}\int_0^{\Phi}\frac{\mathrm{d}\phi}{\sqrt{\sin^2(\Phi/2)-\sin^2(\phi/2)}} = \frac{2\tau_0}{\pi}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}\sqrt{1-A^2u^2}}$$

where $\tau_0 = 2\pi\sqrt{l/g}$. (Use substitution $\sin(\phi/2) = Au$, where $A = \sin(\Phi/2)$.) These integrals cannot be evaluated in terms of elementary functions, but the second integral is a standard integral called the *complete elliptic integral of the first kind,* sometimes denoted $K(A^2)$, whose values can be looked up or calculated with Mathematica's `EllipticK(A^2)`. (b) Use Mathematica (or your favorite software) to make a graph of τ/τ_0 vs. amplitude Φ, for $0 \le \Phi \le 3$ radians and comment. Explain what happens to τ (and why!) as $\Phi \to \pi$.

XC6. Optional/extra-credit. If you have not already done so, do XC5(a). (a) If the amplitude Φ is small, then so is $A = \sin(\Phi/2)$. If the amplitude is very small, we can simply ignore the last square root in the integral in (XC1). Show that this gives the familiar result $\tau = \tau_0 = 2\pi\sqrt{l/g}$. (b) If the amplitude is small but not very small, we can improve on the approximation of part (a). Use the binomial expansion to give the approximation $1/\sqrt{1-A^2u^2} \approx 1 + \frac{1}{2}A^2u^2$ and show that, in this limit, $\tau \approx \tau_0[1 + \frac{1}{4}\sin^2(\Phi/2)]$. (c) What percentage correction does the second term represent for an amplitude of 45°? (The exact answer for $\Phi = 45°$ is $1.040\ \tau_0$ to four significant figures.)

XC7. Optional/extra-credit. A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction, as shown in the left figure below. The beads are released simultaneously from rest (given an infinitesimal kick) at the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > \frac{3}{2}M$, and find the angle θ at which this occurs. [Hint: If $M = 0$, then $\cos\theta = \frac{2}{3}$.] You will receive partial extra-credit if you do the problem assuming $M = 0$, but for full credit, you must account for the mass M of the ring.





XC8. Optional/extra-credit. The right figure above shows a massless wheel of radius R, mounted on a frictionless horizontal axle. A point mass M is glued to the edge of the wheel, and a mass m hangs from a string wrapped around the perimeter of the wheel. (a) Write down the total PE of the two masses as a function of the

angle ϕ. (b) Use this to find the values of m/M for which there are any positions of equilibrium. Describe the equilibrium positions, discuss their stability, and explain your answers in terms of torques. (c) Graph $U(\phi)$ for the cases $m = 0.7M$ and $m = 0.8M$, and use your graphs to describe the behavior of the system if I release it from rest at $\phi = 0$. (If the system oscillates, you **do not** need to find the frequency of oscillation.) (d) Find the critical value of m/M such that if $\frac{m}{M} < (\frac{m}{M})_{\rm crit}$, the system oscillates, while if $\frac{m}{M} > (\frac{m}{M})_{\rm crit}$ it does not (if released from rest at $\phi = 0$).

XC9. Optional/extra-credit. Repeat the calculations of Example 5.3 (page 185) with all the same parameters, but with the initial conditions $x_0 = 2$ and $v_0 = 0$. Graph $x(t)$ for $0 \le t \le 4$ and compare with the graph of Example 5.3. Explain the similarities and differences, e.g. for what region in time do the two graphs differ appreciably?

XC10. Optional/extra-credit. You can make the Fourier series solution for a periodically driven oscillator a bit tidier if you don't mind using complex numbers. Obviously the periodic force of (Eq. 5.90) can be written as $f = \mathrm{Re}(g)$, where the complex function g is $g(t) = \sum_{n=0}^{\infty} f_n e^{in\omega t}$. Show that the real solution for the oscillator's motion can likewise be written as $x = \mathrm{Re}(z)$, where $z(t) = \sum_{n=0}^{\infty} C_n e^{in\omega t}$ and $C_n = f_n/(\omega_0^2 - n^2\omega^2 + 2i\beta n\omega)$. This solution avoids our having to worry about the real amplitude A_n and phase shift δ_n separately.

XC11. Optional/extra-credit. Use the property (4.35) of the gradient to prove the following: (a) The vector ∇f at any point $\boldsymbol{r}$ is perpendicular to the surface of constant f through $\boldsymbol{r}$. (What is $\mathrm{d}f$ for a small displacement $\mathrm{d}\boldsymbol{r}$ that lies in a surface of constant f?) (b) The direction of ∇f at any point $\boldsymbol{r}$ is the direction in which f increases fastest as we move away from $\boldsymbol{r}$. (Choose a small displacement $\mathrm{d}\boldsymbol{r} = \epsilon\boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector and ϵ is fixed and small. Find the direction of $\boldsymbol{u}$ for which the corresponding $\mathrm{d}f$ is maximum, bearing in mind that $\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$.)

XC12. Optional/extra-credit. Section 4.8 claims that a force $\vec{F}(\vec{r})$ that is central and spherically symmetric is automatically conservative. Here are two ways to prove it. (a) Since $\vec{F}(\vec{r})$ is central and spherically symmetric, it must have the form $\vec{F}(\vec{r}) = f(r)\hat{r}$. Using Cartesian coordinates, show that this implies that $\nabla\times\vec{F} = 0$. (b) Even quicker, using the expression given inside the textbook's back cover for $\nabla\times\vec{F}$ in spherical polar coordinates, show that $\nabla\times\vec{F} = 0$.

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