Physics 351, Spring 2018, Homework #4.

Due at start of class, Friday, February 16, 2018

Please write your name only on the VERY LAST PAGE of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

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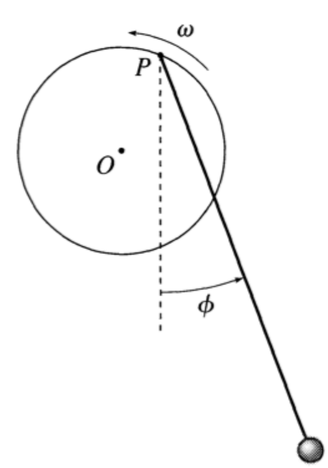
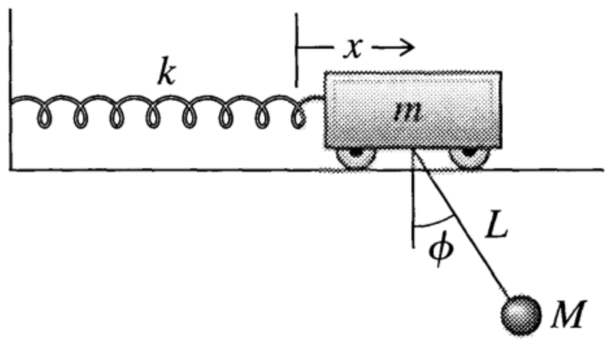
1. Using the usual angle ϕ as a generalized coordinate, write down the Lagrangian for a simple pendulum of length l suspended from the ceiling of an elevator that is accelerating upward with constant acceleration a. (Be careful when writing T. It is probably safest to write the bob's velocity in component form.) Find the Lagrange EOM and show that it is the same as that for a normal, nonaccelerating pendulum, except that g has been replaced by $g + a$. In particular, the angular frequency of small oscillations is $\sqrt{(g+a)/l}$.

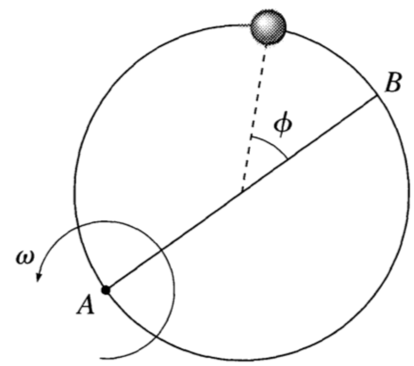
2. Consider a double Atwood machine constructed as follows: A mass $4m$ is suspended from a string that passes over a massless pulley on frictionless bearings. The other end of this string supports a second similar pulley, over which passes a second string supporting a mass of $3m$ at one end and m at the other. Using two suitable generalized coordinates, set up the Lagrangian and find the acceleration of the mass $4m$ when the system is released. Explain why the top pulley rotates even though it carries equal weights on each side.

3. The figure shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the EOM for the angle ϕ. [Hint: Be careful writing down T, the K.E. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case $\omega = 0$.



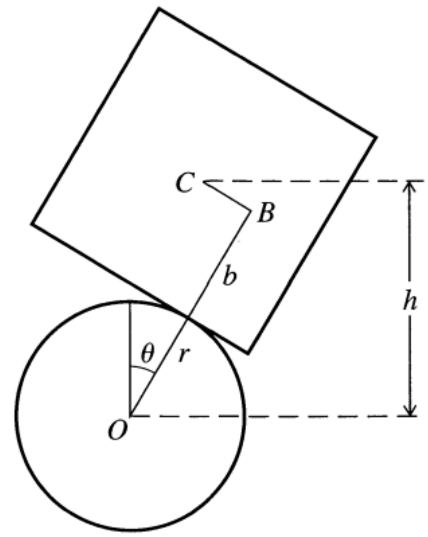
4. A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring (spring constant k), as shown in the left figure below. (a) Write the Lagrangian in terms of the two generalized coordinates x and ϕ, where x is the extension of the spring from its equilibrium length. Find the two Lagrange equations. (They're ugly.) (b) Simplify the equations to the case that both x and ϕ are small. (They're still pretty ugly, and still coupled, but we'll solve them in Chapter 11.)





5. The above-right figure is a bird's-eye view of a smooth horizontal wire hoop that is forced to rotate at a fixed angular frequency ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center with the diameter AB. Find the Lagrangian for this system using ϕ as your generalized coordinate. Use the Lagrange EOM to show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

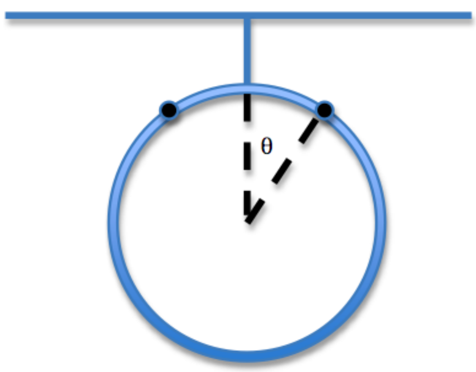
6. Consider the cube balanced on a cylinder, as described in Example 4.7 (page 130). (Immobile cylinder of radius r. Cube of side $2b$ can rock but can't slip. $U(\theta) = mg[(r + b)\cos\theta + r\theta\sin\theta]$.) Assuming that $b < r$, use the Lagrangian approach to find the angular frequency of small oscillations about the top. The simplest procedure is to make the small-angle approximations to $\mathcal{L}$ before you differentiate to get Lagrange's equation. As usual, be careful in writing down the kinetic energy, which is $\frac{1}{2}(mv^2 + I\dot{\theta}^2)$, where v is the speed of the CM and $I = 2mb^2/3$ is the moment of inertia about the CM. The safe way to find v is to write down the coordinates of the CM and then differentiate.

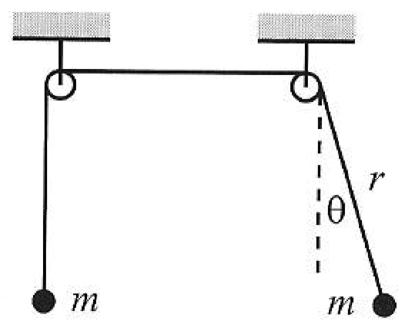


7. A pendulum is made from a massless spring (force constant k and unstretched length l_0) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane. (a) Write down the Lagrangian for the pendulum, using as generalized coordinates the usual angle ϕ and the length r of the spring. (b) Find the two Lagrange equations of the system and interpret them in terms of Newton's second law (Eq. 1.48), $F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$. (c) The equations of part (b) cannot be solved analytically in general, but they *can* be solved for small oscillations. Do this and describe the motion. [Hint: Let l describe the equilibrium length of the spring with the mass hanging from it and write $r = l + \epsilon$. "Small oscillations" involve only small values of ϵ and ϕ, so you can use the small-angle approximations and drop from your equations all terms that involve powers of ϵ or ϕ (or their derivatives) higher than the first power (also products of ϵ and ϕ or their derivatives). This dramatically simplifies and uncouples the equations.

8. A mass m_1 rests on a frictionless horizontal table. Attached to it is a string which runs horizontally to the edge of the table, where it passes over a frictionless, small pulley and down to where it supports a mass m_2. Use as coordinates x and y the distances of m_1 and m_2 from the pulley. These satisfy the constraint equation $f(x, y) = x + y = \text{const}$. Write down the two modified Lagrange equations and solve them (together with the constraint equation) for $\ddot{x}$, $\ddot{y}$, and the Lagrange multiplier λ. Use (Eq. 7.122) (and the corresponding equation in y) to find the tension forces on the two masses. Verify your answers by solving the problem by the elementary Newtonian approach.

9. This is a repeat of HW2/XC7, but now it is a required problem, which you can solve using the Lagrangian approach. But you'll need to use a Lagrange multiplier so that you can solve for the force of constraint imposed by the thread. The trick is to write a Lagrangian having two coordinates: θ (as indicated on the left figure below) and Y, the vertical position of the ring. You then include a Lagrange multiplier term λY to enforce the $Y = 0$ constraint (which also implies $\dot{Y} = 0$ and $\ddot{Y} = 0$), as described in §7.10. "The ring will start to rise" implies $\lambda = 0$, i.e. the tension in the string is zero. There are actually two solutions for $\lambda = 0$, whose meaning you should interpret (even though only one of the two solutions describes the rings' starting to rise). Here's the problem as previously stated: A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction, as shown in the figure. The beads are released simultaneously from rest (given an infinitesimal kick) at the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > \frac{3}{2}M$, and find the angle θ at which this occurs. [Hint: If $M = 0$, then $\cos\theta = \frac{2}{3}$.] One more hint: you will probably find it helpful to use energy conservation (after imposing $Y \equiv 0$) to write $\dot{\theta}$ in terms of $\cos\theta$.





10. Two equal masses m, connected by a massless string, hang over two pulleys (of negligible size), as shown in the above-right figure. The left mass moves in a vertical line, but the right mass is free to swing back and forth in the plane of the masses and pulleys. Find the EOM for r and θ, as shown. Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude ϵ ($\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?

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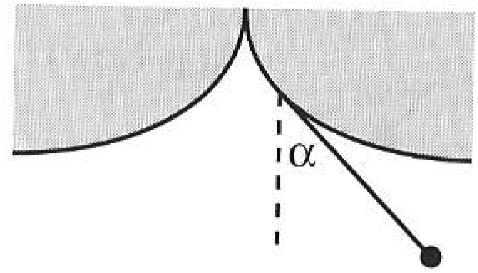
XC1. Optional/extra-credit. Two equal masses, $m_1 = m_2 = m$, are joined by a massless string of length L that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second hangs below the table and moves up and down in a vertical line. (a) Assuming the string remains taut, write down the Lagrangian for the system in terms of the polar coordinates (r, ϕ) of the mass on the table. (b) Find the two Lagrange equations and interpret the ϕ equation in terms of the angular momentum ℓ of the first mass. (c) Express $\dot{\phi}$ in terms of ℓ and eliminate $\dot{\phi}$ from the r equation. Now use the r equation to find the value $r = r_0$ at which the first mass can move in a circular path. Interpret your answer in Newtonian terms. (d) Suppose the first mass is moving in this circular path and is given a small radial nudge. Write $r(t) = r_0 + \epsilon(t)$ and rewrite the r equation in terms of $\epsilon(t)$ dropping all powers of $\epsilon(t)$ higher than linear. Show that the circular path is stable and that $r(t)$ oscillates sinusoidally about r_0. What is the frequency of its oscillations?

XC2. Optional/extra-credit. In Problem 3, one might expect that the rotation of the wheel would have little effect on the pendulum, provided the wheel is small and rotates slowly. (a) Verify this expectation by solving the EOM numerically, with the following numbers: Take $g = l = 1$ (so the natural frequency $\sqrt{g/l}$ is also 1). Take $\omega = 0.2$, so that the wheel's rotational frequency is small compared to the natural frequency of the pendulum; and take the radius $R = 0.2$, significantly less than the length of the pendulum. As initial conditions take $\phi = 0.2$ and $\dot{\phi} = 0$ at $t = 0$, and make a graph of your solution $\phi(t)$ for $0 < t < 20$. Your graph should resemble the sinusoidal oscillations of an ordinary simple pendulum. Does the period look correct? (b) Now graph $\phi(t)$ for $0 < t < 100$ and notice that the rotating support does make a small difference, causing the amplitude of the oscillations to grow and shrink periodically. Comment on the period of these small fluctuations.

XC3. Optional/extra-credit. In Example 7.7 (page 264), we saw that the bead on a spinning hoop can make small oscillations about its nonzero stable equilibrium points that are approximately sinusoidal, with frequency (as in Eq. 7.80) $\Omega' = \sqrt{\omega^2 - g^2/(\omega R)^2}$. Investigate how good this approximation is by solving the EOM (Eq. 7.73) numerically and then plotting both your numerical solution and the approximate solution $\theta(t) = \theta_0 + A\cos(\Omega' t - \delta)$ on the same graph. Use the following numbers: $g = R = 1$ and $\omega^2 = 2$, and initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = \theta_0 + \epsilon_0$, where $\epsilon_0 = 1°$. Repeat with $\epsilon_0 = 10°$. Comment on your results.

XC4. Optional/extra-credit. The standard pendulum frequency $\sqrt{g/\ell}$ holds only for small oscillations. The frequency becomes smaller as the amplitude grows. It turns out that if you want to build a pendulum whose frequency is independent of the amplitude, you should hang it from the cusp of a cycloid of a certain size, as shown in the figure. As the string wraps partially around the cycloid, the effect is to decrease the length of string in the air, which in turn increases the frequency back up to a constant value. In more detail:



A cycloid is the path taken by a point on the rim of a rolling wheel. The upside-down cycloid in the figure can be parametrized by $(x, y) = R(\theta - \sin\theta, -1 + \cos\theta)$, where $\theta = 0$ corresponds to the cusp. Consider a pendulum of length $4R$ hanging from the cusp, and let α be the angle the string makes w.r.t. vertical, as shown.

(a) In terms of α, find the value of the parameter θ associated with the point where the string leaves the cycoid.

(b) In terms of α, find the length of string touching the cycoid.

(c) In terms of α, find the Lagrangian.

(d) Show that the quantity $\sin\alpha$ undergoes simple harmonic motion with frequency $\sqrt{g/(4R)}$, independent of the amplitude.

(e) In place of parts (c) and (d), solve the problem again by using $F = ma$. This actually gives a much quicker solution!

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