Physics 351, Spring 2018, Homework #10.

Due at start of class, Friday, April 6, 2018

Please write your name on the $\boxed{\text{LAST PAGE}}$ of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

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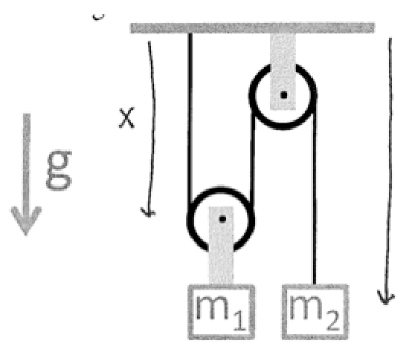
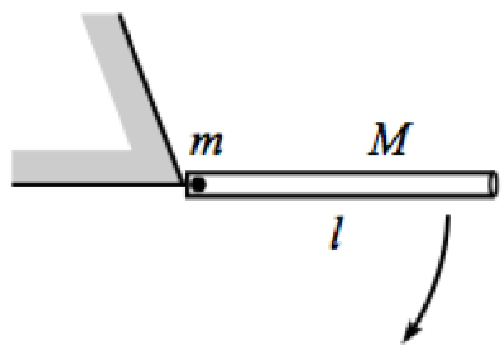
1. Here is a problem that is a weird variation of an Atwood's machine. The goal is to find the accelerations of m_1 and m_2, as shown in the left figure below. Also find the tension in the string! Assume that the pulleys are massless and frictionless, so that the tension in the string is constant. The preferred way to solve this problem is in the Lagrangian framework, using a Lagrange multiplier; the constraint can be expressed at $2x + y = L$, the total length of the string.





2. A tube of mass M and length ℓ is free to swing by a pivot at one end. (Use the moment of inertia of a uniform thin rod rotating about one end.) A mass m is positioned inside the tube at this end. The tube is held horizontal and then released. (See above-right figure.) Let θ be the angle of the tube w.r.t. the horizontal, and let x be the distance the mass has traveled along the tube. Find the Lagrange equations of motion for θ and x, then write them in terms of θ and $\eta \equiv x/\ell$ (the fraction of the distance along the tube). These equations can only be solved numerically, and you must pick a numerical value for the ratio $r \equiv m/M$ in order to do this. Use Mathematica (or your favorite alternative) to find the value of η when the tube is vertical ($\theta = \pi/2$). Give this value of η for a few values of r.

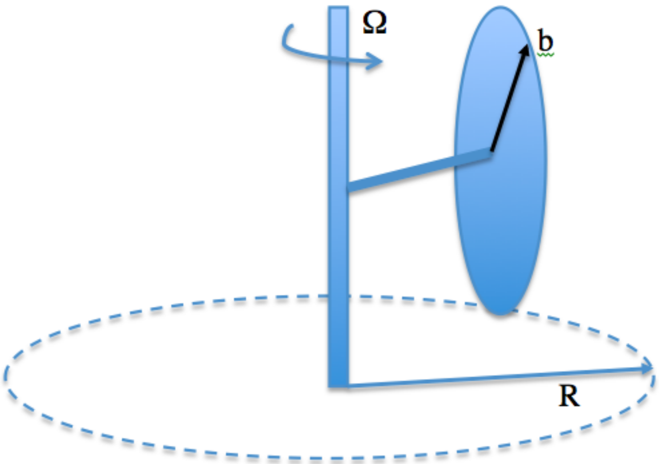
3. [This problem can take a very long time to puzzle over, but in the end you'll understand much better what Taylor means by Ω_s and Ω_b.] We saw in §10.8 that in the free precession of an axially symmetric body ($\lambda_1 = \lambda_2$) the three vectors $\hat{\boldsymbol{e}}_3$ (the symmetry axis), $\boldsymbol{\omega}$, and $\boldsymbol{L}$ lie in a plane. As seen in the body frame, $\hat{\boldsymbol{e}}_3$ is fixed, and $\boldsymbol{\omega}$ and $\boldsymbol{L}$ precess around $\hat{\boldsymbol{e}}_3$ with angular velocity $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1$. As seen in the space frame, $\boldsymbol{L}$ is fixed, and $\boldsymbol{\omega}$ and $\hat{\boldsymbol{e}}_3$ precess around $\boldsymbol{L}$ with angular frequency Ω_s. In this [very lengthy!] problem you will find three equivalent expressions for Ω_s. (a) Argue that $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. [Remember that relative angular velocities add like vectors.] (b) Bearing in mind that $\boldsymbol{\Omega}_b$ is parallel to $\hat{\boldsymbol{e}}_3$, prove that $\Omega_s = \omega \sin\alpha / \sin\theta$, where α is the angle between $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{\omega}$, and θ is the angle between $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{L}$. (See Figure 10.9.) [Hint: look at the equation from part (a) and consider the component perpendicular to $\hat{\boldsymbol{e}}_3$.] (c) Thence prove that

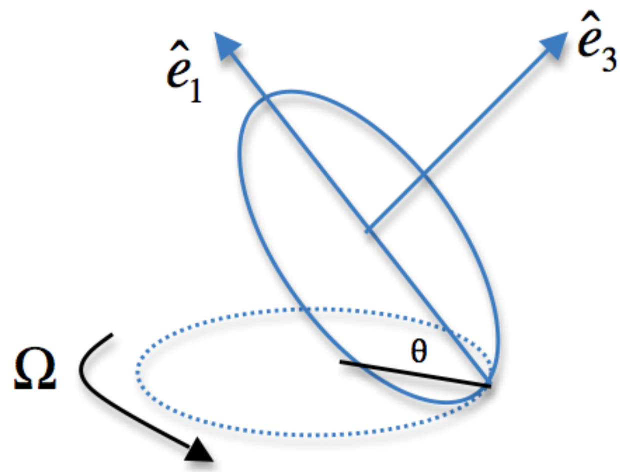
$$\Omega_s = \omega \frac{\sin\alpha}{\sin\theta} = \frac{L}{\lambda_1} = \omega \, \frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}$$

4. Consider the rapid steady precession of a symmetric top predicted in connection with (Eq. 10.112). (a) Show that in this motion the angular momentum $\boldsymbol{L}$ must be very close to the vertical. [Hint: Use (Eq. 10.100) to write down the horizontal component L_{hor} of $\boldsymbol{L}$. Show that if $\dot{\phi}$ is given by the right side of (Eq. 10.112), L_{hor} is exactly zero.] (b) Use this result to show that the rate of precession Ω given in (Eq. 10.112) agrees with the free precession rate Ω_s found in (Eq. 10.96).

5. In the discussion of steady precession of a top in §10.10, the rates Ω at which steady precession can occur were determined by the quadratic equation (Eq. 10.110). In particular, we examined this equation for the case that ω_3 is very large. In this case you can write the equation as $a\Omega^2 + b\Omega + c = 0$ where b is very large. (a) Verify that when b is very large, the two solutions of this equation are approximately $-c/b$ (which is small) and $-b/a$ (which is large). What precisely does the condition "b is very large" entail? (You should find a dimensionless ratio $\gg 1$.) (b) Verify that these give the two solutions claimed in (Eq. 10.111) and (Eq. 10.112).

6. [Here's a problem from a final exam 4 or 5 years ago.] In a "rolling mill," grain is ground by a disk-shaped millstone that rolls in a circle on a flat surface and is driven by a vertical shaft. Assume that the millstone is a uniform disk of radius b and negligible thickness. ($\lambda_3 = \frac{1}{2}mb^2$. What is $\lambda_1 = \lambda_2$, given that this object is planar?) Also assume that the wheel cannot tip, so it always remains perpendicular to the ground. The wheel rolls without slipping along a circle of radius R with angular velocity Ω as indicated in the figure below (left). Show that the normal force that the ground exerts on the wheel is $Mg + \frac{1}{2}Mb\Omega^2$. Because of the angular momentum of the millstone, the contact force with the surface can be much larger than the weight of the wheel, which is what makes this an effective way to grind grain.

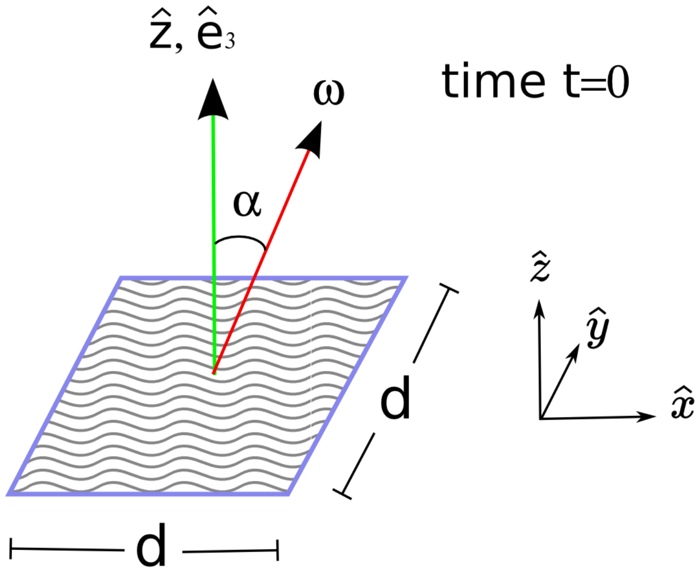




7. When you spin a coin around a vertical diameter on a table, it will lose energy and go into a wobbling motion, whose frequency increases as the coin's angle w.r.t. horizontal decreases. Consider the moment when the coin makes an angle θ w.r.t. the horizontal surface of the table. Assume that the CM of the coin is motionless and that the contact point moves along a circle on the table, as shown in the left figure above (right). Let the radius of the coin be R, and let Ω be the angular velocity of the motion of the contact point. Assume that the coin rolls without slipping. (a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega \sin\theta\, \hat{\boldsymbol{e}}_1$, where $\hat{\boldsymbol{e}}_1$ points upward along the coin, diametrically away from the contact point. (b) Show that $\Omega = 2\sqrt{g/(R\sin\theta)}$.

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.

(a) Show that the inertia tensor has the form $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0. (b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. (c) Draw a sketch showing the vectors $\hat{\boldsymbol{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$) than it is for the American-football-like ("prolate") object ($\lambda_3 < \lambda_1$) drawn on Taylor's page 400. (d) Draw and label the "body cone" and the "space cone" on your sketch. (e) Calculate the precession frequencies $\Omega_{\rm body}$ and $\Omega_{\rm space}$. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. (You puzzled through these directions when you solved problem 3.) (f) You argued in problem 3 that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$. (g) Find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{e}}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α (dropping terms $\mathcal{O}(\alpha^2)$ and higher). (h) When is this maximum deviation first reached? (i) As a check, verify (for the $\alpha \ll 1$ limit) that Feynman indeed misremembered which way the factor of two had gone in this anecdote about a plate tossed through the air in a Cornell cafeteria: `positron.hep.upenn.edu/p351/feynman`



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XC1. Optional/extra-credit. An important special case of the motion of a symmetric top occurs when it spins about a vertical axis. Analyze this motion as follows: (a) By inspecting the effective PE (Eq. 10.114), show that if at any time $\theta = 0$, then L_3 and L_z must be equal. (b) Set $L_z = L_3 = \lambda_3\omega_3$ and then make a Taylor expansion of $U_{\rm eff}(\theta)$ about $\theta = 0$ to terms of order θ^2. (c) Show that if $\omega_3 > \omega_{\rm min} = 2\sqrt{MgR\lambda_1/\lambda_3^2}$, then the position $\theta = 0$ is stable, but if $\omega_3 < \omega_{\rm min}$ it is unstable. (In practice, friction slows the top's spinning. Thus with ω_3 sufficiently fast, the vertical top is stable, but as it slows down the top will eventually lurch away from the vertical when ω_3 reaches $\omega_{\rm min}$.)

XC2. Optional/extra-credit. [Computer] The nutation of a top is controlled by the effective potential energy (Eq. 10.114). Make a graph of $U_{\rm eff}(\theta)$ as follows: (a) First, since the second term of $U_{\rm eff}(\theta)$ is a constant, you can ignore it. Next, by choice of your units, you can take $MgR = 1 = \lambda_1$. The remaining parameters L_z and L_3 are genuinely independent parameters. To be definite set $L_z = 10$ and $L_3 = 8$ and plot $U_{\rm eff}(\theta)$ as a function of θ. (b) Explain clearly how you would use your graph to determine the angle θ_0 at which the top could precess steadily with $\theta =$ constant. Find θ_0 to three significant figures. (c) Find the rate of this steady precession, $\Omega = \dot{\phi}$, as given by (Eq. 10.115). Compare with the approximate value of Ω given by (Eq. 10.112).

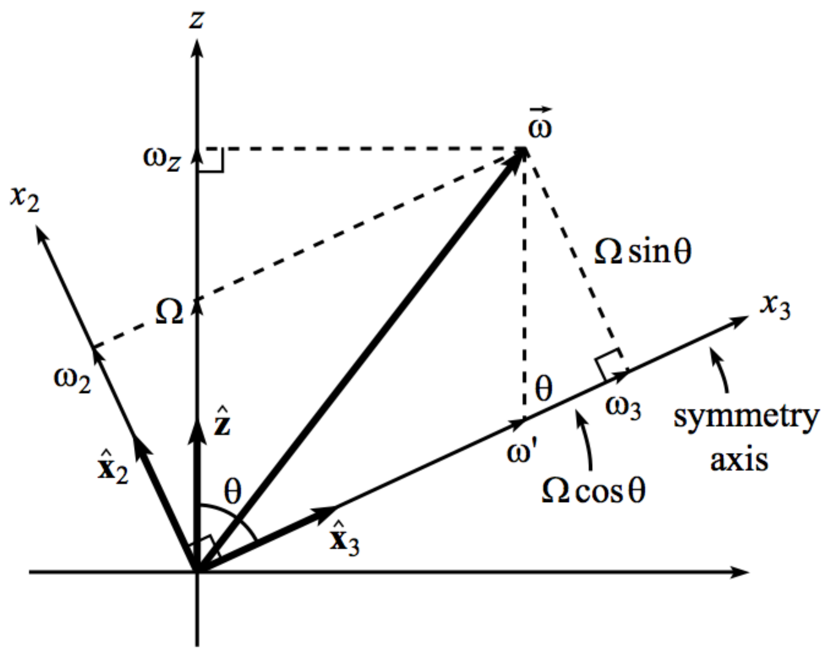
XC3. Optional/extra-credit. Do Taylor's problem 10.33 (page 412), which is too long to retype here. It involves deriving expressions for T and for $\boldsymbol{L}$.

XC4. Optional/extra-credit. Consider a rotating reference frame such as a frame fixed on the earth's surface. A particle is thrown vertically up with initial speed v_0, reaches a maximum height, and falls back to the ground. Show that the Coriolis deflection when it reaches the ground is four times as large as and in the opposite direction from the Coriolis deflection when it is dropped from rest at the same maximum height. Can you explain why?

XC5. Optional/extra-credit. Assume that a piece of toast is a rigid uniform square of side length ℓ. You butter the toast and then drop it from a height H above a table; the table is a height h above the floor. The toast starts off parallel to the table, and as it falls, it clips the edge of the table and collides elastically, causing the toast to start to rotate. You want to find the value of H, in terms of h and ℓ, that leads to the sad situation in which the toast makes exactly one-half revolution and lands on the floor butter-side-down. Show that $H = \frac{\pi^2\ell^2}{6\,(6h-\pi\ell)}$. [Hint: with a clever choice of origin, you can argue that the angular momentum of the toast is conserved during the collision with the table.]

XC6. Optional/extra-credit. (a) A small ball of radius r and uniform density rolls without slipping at the bottom of a fixed cylinder of radius $R \gg r$. Show that the frequency of small oscillations is $\omega = \sqrt{\frac{5g}{7R}}$. [You'll need $I = \frac{2}{5}Mr^2$ for a uniform sphere about an axis through its center.] (b) Generalize your result for the case where the sphere's density is not uniform (but is still spherically symmetric), so its moment of inertia is given by $I = \beta M r^2$.

XC7. Optional/extra-credit. Consider a top made of a wheel with all its mass on the rim. A massless rod (perpendicular to the plane of the wheel) connects the CM to a pivot. Initial conditions have been set up so that the top undergoes precession, with the rod always horizontal. In the language of the figure below (Morin's Fig. 9.30), we may write the angular velocity of the top as $\boldsymbol{\omega} = \Omega\hat{\boldsymbol{z}} + \omega'\hat{\boldsymbol{x}}_3$ (where $\hat{\boldsymbol{x}}_3 = \hat{\boldsymbol{e}}_3$ is horizontal here). Consider things in the frame rotating around the $\hat{\boldsymbol{z}}$ axis with angular speed Ω. In this frame, the top spins with angular speed ω' around its *fixed* symmetry axis. Therefore, in this frame we must have $\boldsymbol{\tau} = 0$, because $\boldsymbol{L}$ is constant. Verify explicitly that $\boldsymbol{\tau} = 0$ (calculated w.r.t. the pivot) in this rotating frame (you will need to find the relation between ω' and Ω). In other words, show that the torque due to gravity is exactly canceled by the torque due to the Coriolis force (you can quickly show that the centrifugal force provides no net torque). Remember that HW09/XC3 implies a Coriolis torque of magnitude $m\omega'\Omega r^2$.



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