Physics 351, Spring 2018, Homework #11.

Due at start of class, Friday, April 13, 2018

Please write your name on the LAST PAGE of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

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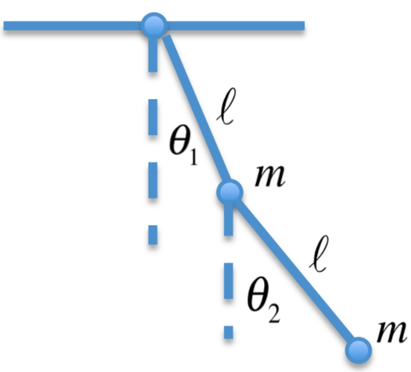
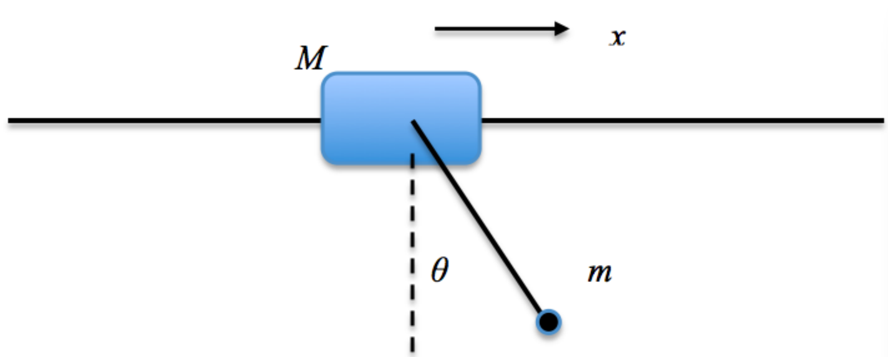
1. [Here's a problem that appeared on a midterm exam 4 or 5 years ago; the last two parts were extra-credit on the exam, but you have to solve them!] Consider the double pendulum consisting of two bobs confined to move in a plane. The rods are of equal length ℓ, and the bobs have equal mass m. The generalized coordinates used to describe the system are θ_1 and θ_2, the angles that the rods make with the vertical (see left figure below). (a) Write the Lagrangian for the system. (This could be an opportunity to practice writing $(\boldsymbol{v}_1 + \boldsymbol{v}_2)^2 = v_1^2 + v_2^2 + 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$.) (b) Next, simplify your Lagrangian from part (a) by assuming that angles θ_1 and θ_2 are both small. Keep terms up to second order in the angles, the angular velocities, and their products. (c) Find the two Lagrange equations of motion, which will be a set of coupled, linear differential equations. (d) Solve the equations of motion (e.g. using the techniques of Chapter 11).





2. [Here's another problem from a midterm exam 4 or 5 years ago; the last part was extra-credit on the exam.] A block of mass M moves on a frictionless horizontal rail. A pendulum of length L and mass m hangs from the block. (See right figure above.) Let x be the displacement of the block, and let θ be the angular displacement of the pendulum w.r.t. the vertical. (a) Write the Lagrangian for the system. (Another possible opportunity to write $(\boldsymbol{v}_1 + \boldsymbol{v}_2)^2 =$

$v_1^2 + v_2^2 + 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$?) (b) Which of the coordinates is ignorable ("cyclic")? What is the associated conserved quantity? This is an example of what conservation law? (c) Find the Lagrange equations of motion for the system. (d) Simplify the equations of motion found in part (c) for the case of small oscillations (where you can discard any terms of second order or higher in the displacements, velocities, or their products). (e) Solve the system of differential equations from part (d) and determine the most general motion of the system. (Your solution should have four arbitrary constants. Using the results of part (b) should help you to simplify the problem.)

3. *Hamiltonian treatment of the symmetric top.* [Here's a problem that appeared on a previous year's final exam.] Consider a symmetric top ($\lambda_1 = \lambda_2$) whose tip has a fixed location in space. Using the Euler angles ϕ, θ, and ψ (whose detailed definitions are not needed for you to solve this problem) to represent the top's orientation, the top's Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\lambda_1 \dot{\phi}^2 \sin^2\theta + \frac{1}{2}\lambda_1 \dot{\theta}^2 + \frac{1}{2}\lambda_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

where M is the mass of the top and R is the distance from the contact point to the top's CoM. λ_3 is the moment of inertia for the top's symmetry axis, and λ_1 is the moment of inertia for the other two principal axes. (a) Calculate the three generalized momenta, p_ϕ, p_θ, and p_ψ. (b) The simplest way to construct the Hamiltonian is to realize that the coordinates are natural, so $H = T + U$. Use this to show that the Hamiltonian is given by

$$H = \frac{(p_\phi - p_\psi \cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3} + MgR\cos\theta$$

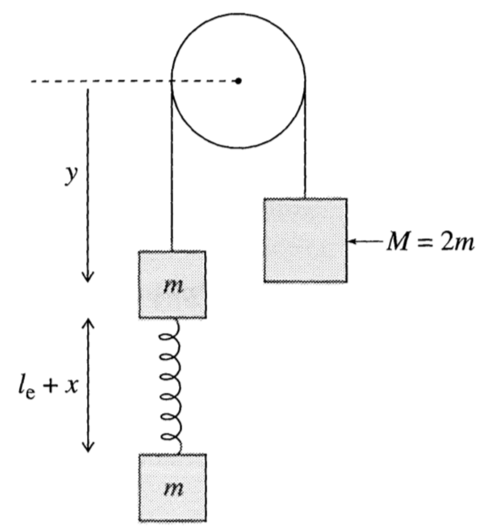
(c) Two of the Euler-angle coordinates are ignorable. Which ones? The corresponding generalized momenta are constant. Use this to show that the Hamiltonian can be written as

$$H = \frac{p_\theta^2}{2\lambda_1} + U_{\text{eff}}(\theta)$$

What is the effective potential energy U_{eff} for this system?

4. Two masses m_1 and m_2 are joined by a massless spring (force constant k and natural length l_0) and are confined to move in a frictionless horizontal plane, with CM and relative positions $\boldsymbol{R}$ and $\boldsymbol{r}$ as defined in §8.2. (a) Write down the Hamiltonian $\mathcal{H}$ using as generalized coordinates X, Y, r, ϕ, where (X, Y) are the rectangular components of $\boldsymbol{R}$, and (r, ϕ) are the polar coordinates of $\boldsymbol{r}$. Which coordinates are ignorable and which are not? Explain. (b) Write down the 8 Hamilton equations of motion. (c) Solve the r equations for the special case that $p_\phi = 0$ and describe the motion.

5. Consider the modified Atwood machine shown in the figure below. The two weights on the left have equal masses m and are connected by a massless spring of Hooke's-law constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$, where l_e is the equilibrium length (with all the weights in position and M held stationary). (a) Show that the total potential energy (spring plus gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that we can take to be zero). (b) Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$, and write down the Hamiltonian. Show that the coordinate y is ignorable. (c) Write down the four Hamilton equations an solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m down a distance x_0, and at $t = 0$ you let go of both masses. [Hint: Write down the initial values of x, y, and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.] Describe the motion. In particular, find the frequency with which x oscillates.



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XC1. Optional/extra-credit. All of the examples in Taylor's Chapter 13 and all of the problems (except this one) treat forces that come from a potential energy $U(\boldsymbol{r})$ [or occasionally $U(\boldsymbol{r}, t)$]. However, the proof of Hamilton's equations given in §13.3 applies to any system for which Lagrange's equations hold, and this can include forces not derivable from a potential energy. An important example of such a force is the magnetic force on a charged particle. (a) Use the Lagrangian (Eq. 7.103) to show that the Hamiltonian for a charge q in an electromagnetic field is $\mathcal{H} = (\boldsymbol{p} - q\boldsymbol{A})^2/(2m) + qV$. (This Hamiltonian plays an important role in the quantum mechanics of charged particles.) (b) Show that Hamilton's equations are equivalent to the familiar Lorentz force equation $m\ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

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