Physics 351, Spring 2018, Homework #12.

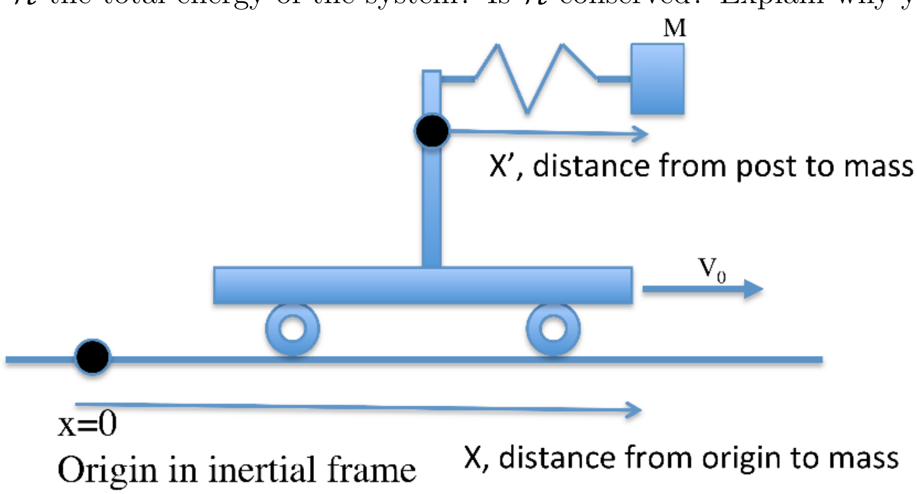
Due at start of class, Friday, April 20, 2018
(but it's OK with me if you turn it in on Monday, April 23)

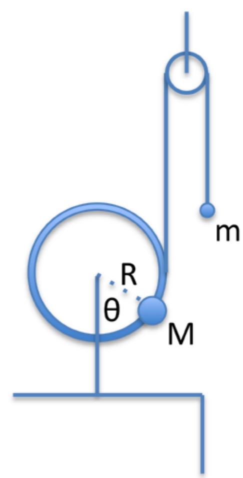
Please write your name on the LAST PAGE of your homework submission, so that we don't notice whose paper we're grading until we get to the very end.

When you finish this homework, remember to tell me how the homework went for you, by visiting the feedback page at

`positron.hep.upenn.edu/q351`

1. Consider a point mass M attached to a spring (force constant k), whose other end is attached to a massless cart that is moved by an external device at the constant speed v_0. You will consider $\mathcal{H}$ for the system using two different generalized coordinates. (a) First, consider the system using the variable x, which is referenced to a fixed origin. (See left figure below.) Write down $\mathcal{L}$ and find the Lagrange EOM for x. Now construct $\mathcal{H}$. Does $\mathcal{H}$ equal the total energy of the system? Is $\mathcal{H}$ conserved? Explain why your answers are OK. (b) Second, analyze the system again using the "relative coordinate" x', which is the displacement of the point mass relative to the cart. (x' is measured from the equilibrium position of the mass M.) Write $\mathcal{L}$ and find the Lagrange EOM for x'. Construct $\mathcal{H}$. ($\mathcal{H}$ is different from what you found in the first part because you are using a different coordinate.) Is $\mathcal{H}$ the total energy of the system? Is $\mathcal{H}$ conserved? Explain why your answers are OK.





2. [This (Lagrangian) problem is from an exam from 4 or 5 years ago.] A mass M is attached to a massless hoop of radius R that lies in a vertical plane and is free to rotate about its fixed center. M is tied to a string that winds part way around the hoop and then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string. (See above-right figure.) Find the EOM for the angle θ of rotation of the hoop, where $\theta = 0$ would put M directly below the center of the hoop. What is the frequency of small oscillations about the equilibrium angle θ_0? Assume that m moves only vertically and that $M > m$.

3. Consider a function $f(q,p)$ of the coordinates q and p. Use Hamilton's equations to show that the time derivative of f can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q}$$

Note: this combination of partial deriviates comes up often enough to warrant a name. The *Poisson bracket* of two functions, f_1 and f_2, is defined to be

$$\{f_1, f_2\} \equiv \frac{\partial f_1}{\partial q}\frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial q}$$

With this definition, the time derivative of f takes the nice compact form, $\mathrm{d}f/\mathrm{d}t = \{f, \mathcal{H}\}$. (More generally, for $f(q,p,t)$, $\mathrm{d}f/\mathrm{d}t = \{f,\mathcal{H}\} + \partial f/\partial t$.) Once you've seen *commutators* in quantum mechanics, you may enjoy this analogy: $[A,B] = i\hbar\,\{A,B\}$. To give just one of many examples, the classical result $\{x, p_x\} = 1$ has the QM analogy $[x, p_x] = i\hbar$.

4. Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw there that there have to be maximum and minimum heights z_{max} and z_{min}, beyond which the mass cannot stray. When z is a maximum or minimum, it must be that $\dot{z} = 0$. Show that this can happen if and only if the conjugate momentum $p_z = 0$, and use the equation $\mathcal{H} = E$, where $\mathcal{H}$ is the Hamiltonian function (Eq. 13.3), to show that, for a given energy E, this occurs at exactly two values of z. [Hint: Write down the function $\mathcal{H}$ for the case that $p_z = 0$ and sketch its behavior as a function of z for $0 < z < \infty$. How many times can this function equal any given E?] Use your sketch to describe the motion of the mass.

5. Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw that there are solutions for which the mass remains at the fixed height $z = z_0$, with fixed angular velocity $\dot{\phi}_0$ say. (a) For any chosen value of p_ϕ, use (Eq. 13.34) to get an equation that gives the corresponding value of the height z_0. (b) Use the equations of motion to show that this motion is stable. That is, show that if the orbit has $z = z_0 + \epsilon$ with ϵ small, then ϵ will oscillate about zero. (c) Show that the angular frequency of these oscillations is $\omega = \sqrt{3}\,\dot{\phi}_0 \sin\alpha$, where α is the half angle of the cone ($\tan\alpha = c$ where c is the constant in $\rho = cz$). (d) Find the angle α for which the frequency of oscillation ω is equal to the orbital angular velocity $\dot{\phi}_0$, and describe the motion for this case.

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XC00. Optional/extra-credit. If there are extra-credit problems from earlier homework assignments that you didn't have time to do sooner, you can feel free to turn them in with HW12 for full credit. Just clearly indicate which problem you're solving.

XC1. Optional/extra-credit. You can do any subset you wish of Taylor's 12.6, 12.7, 12.8, 12.9, 12.10, 12.14, 12.15, 12.32, 12.33, 12.34 (all of which involve some sort of modeling of the DDP or the Logistic Map using Mathematica) and turn them in for extra credit. (Each one counts as an extra-credit problem. Once you've done one of them, it should be easy to do several more.)

XC2. Optional/extra-credit. Remember that if you want to, you can read Chapter 14 (collisions/scattering) for extra credit. If you do so, you simply need to answer the online reading questions (for date 2018-04-18, i.e April 18) to collect your extra credit. If you do read Chapter 14, you can also solve any Chapter 14 problems you like for extra credit (even easy ones). Just turn them in with this homework and clearly indicate which problem you're solving.

XC3. Optional/extra-credit. Remember that if you want to, you can read Chapter 16 (continuum mechanics) for extra credit. If you do so, you simply need to answer the online reading questions (for date 2018-04-27, i.e April 27) to collect your extra credit. You can also do any Chapter 16 problems you like for extra credit (even easy ones). Just turn them in with this homework and clearly indicate which problem you're solving.

XC4. Optional/extra-credit. Since we didn't spend much time on Chapter 11 (coupled oscillators), you can do any ** or *** problems you wish from Chapter 11 for extra credit. Just turn them in with this homework and clearly indicate which problem you're solving.

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