Physics 351, Spring 2015, Midterm Exam.

This closed-book exam has (only) 15% weight in your course grade. You can use one sheet of your own hand-written notes. Please show your work on these sheets. The back side of each page is blank, so you can continue your work on the reverse side if you run out of space. Try to work in a way that makes your reasoning obvious to me, so that I can give you credit for correct reasoning even in cases where you might have made a careless error. Correct answers without clear reasoning may not receive full credit. [**This term you get a 3×5 card for notes, not a whole sheet of paper**].

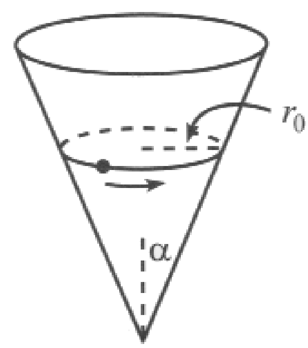
The last page of the exam contains a list of equations that you might find helpful, to complement your own sheet of notes. You can detach it now if you like, before we begin.

The exam contains three questions. The first question is worth 50%. The second and third questions are worth 25% each.

Because I believe that most of the learning in a physics course comes from your investing the time to work through homework problems, these exam problems are similar or identical to problems that you have already solved. The only point of the exams, in my opinion, is to motivate you to take the weekly homework seriously. So you should find nothing surprising in this exam.

[**The original exam was spaced out to 7 pages total. In this edited copy, I've removed the extra space.**]

Problem 1. (50%)
A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the figure. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone.



(a) Write down the Lagrangian, then find the Lagrange equations of motion for ρ and for ϕ.

(b) One of these equations of motion identifies a conserved quantity, which you can plug into the other equation of motion. Name the conserved quantity. What feature of the Lagrangian (sometimes stated as a property of the corresponding coordinate) led us directly to this conserved quantity?

(c) If the particle moves in a circle of radius $\rho = r_0$, what is the angular frequency ω of this motion? Express ω in terms of g, r_0, and α.

(d) If the particle is then perturbed slightly from this circular motion, what is the angular frequency Ω of the oscillations about the radius $\rho = r_0$? Express Ω in terms of ω and α.

(e) Very briefly describe (no math needed) the shape of the orbit if $\Omega = \omega$. What would be the significance of Ω/ω being a rational number? What if Ω/ω were an irrational number?

Problem 2. (25%)
A particle travels in a **parabolic** orbit in a planet's gravitational field and skims the surface at its closest approach. The (spherical) planet has uniform mass density ρ. Relative to the center of the planet, what is the angular velocity of the particle as it skims the surface? (Assume that the mass of the particle is negligible in comparison to the mass of the planet.)

Problem 3. (25%)
At a polar angle θ (colatitude), a projectile is fired due north with initial velocity v_0 at an inclination angle α above the ground. Working to first order in the earth's rotational velocity Ω, show that the eastward deflection due to the Coriolis force is (as a function of the time t since the projectile was fired)

$$x(t) = \Omega v_0(\cos\alpha\cos\theta - \sin\alpha\sin\theta)t^2 + \frac{1}{3}\Omega g t^3 \sin\theta$$

[Assume that air resistance is negligible and that $\boldsymbol{g}$ is a constant throughout the flight. I recommend first neglecting the Coriolis force and writing the "zeroth order" $y(t)$ (northward) and $z(t)$ (upward) for ordinary projectile motion; then use this zeroth-order trajectory to calculate the first-order Coriolis deflection, after evaluating $\boldsymbol{\Omega}$ in terms of the local $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ axes.]

Possibly useful equations.

$$U(r) = -\frac{\gamma}{r} \qquad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \qquad r(\phi) = \frac{c}{1+\epsilon\cos\phi} \qquad c = \frac{\ell^2}{\gamma\mu}$$

Remember that the four different orbit shapes are circle, ellipse, parabola, hyperbola.

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega}\times\boldsymbol{Q}$$

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\,\dot{\boldsymbol{r}}\times\boldsymbol{\Omega} + m\,(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$$