Physics 351, Spring 2017, Midterm Exam.

This closed-book exam has (only) 15% weight in your course grade. You can use one 3×5 card of your own hand-written notes. Please show your work on these sheets. The back side of each page is blank, so you can continue your work on the reverse side if you run out of space. Try to work in a way that makes your reasoning obvious to me, so that I can give you credit for correct reasoning even in cases where you might have made a careless error. Correct answers without clear reasoning may not receive full credit.

Write your name on this page now. You should leave the exam closed until everyone is ready to begin. The exam contains three questions. The first and third questions are worth 35% each. The second question is worth 30%.

Because I believe that most of the learning in a physics course comes from your investing the time to work through homework problems, these exam problems are similar or identical to problems that you have already solved. The only point of the exams, in my opinion, is to motivate you to take the weekly homework seriously. So you should find nothing surprising in this exam.

Possibly useful equations.

$$U(r) = -\frac{\gamma}{r} \qquad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \qquad r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \qquad c = \frac{\ell^2}{\gamma\mu}$$

Remember that the four different orbit shapes are circle, ellipse, parabola, hyperbola.

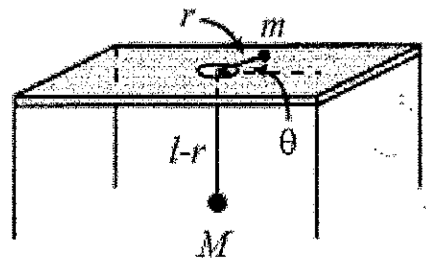
$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega}\times\boldsymbol{Q}$$

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\,\dot{\boldsymbol{r}}\times\boldsymbol{\Omega} + m\,(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \boldsymbol{r}_1 = \boldsymbol{R}_{\text{cm}} + \frac{m_2}{M}\boldsymbol{r} \qquad \boldsymbol{r}_2 = \boldsymbol{R}_{\text{cm}} - \frac{m_1}{M}\boldsymbol{r} \qquad \mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$

Name: BILL

Problem 1. (35%)
A mass m is free to slide on a frictionless table and is connected, via a string that passes through a hole in the table, to a mass M that hangs below. Assume that M moves in a vertical line only, and assume that the string always remains taut.



(a) Write down the Lagrangian, then find the Lagrange equations of motion for r and for θ.

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mgr$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) \Rightarrow \boxed{mr^2\dot{\theta} = L}$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - Mg = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{d}{dt}\left((m+M)\dot{r}\right) = (m+M)\ddot{r}$$

$$\boxed{(m+M)\ddot{r} = mr\dot{\theta}^2 - Mg}$$

(b) One of these equations of motion identifies a conserved quantity (a constant of the motion), which you can plug into the other equation of motion. Name the conserved quantity. What feature of the Lagrangian (sometimes stated as a property of the corresponding coordinate) led us directly to this conserved quantity?

$L \equiv$ z component of angular momentum is a constant of the motion, because $\mathcal{L}$ is independent of θ, i.e. θ is an "ignorable" coordinate.

$mr^2\dot{\theta} \equiv L$ is a constant of the motion.

(c) If the mass m moves in a circle of radius $r = r_0$, what is the angular frequency ω of this motion? Express ω in terms of m, M, g, and r_0.

if $r \equiv r_0$ then $\ddot{r} = 0 \Rightarrow m r_0 \dot{\theta}^2 = Mg$

$$\Rightarrow \omega^2 = \frac{Mg}{mr_0} \Rightarrow \boxed{\omega = \sqrt{\frac{Mg}{mr_0}}}$$

(notice same units as familiar $\sqrt{\frac{g}{\ell}}$ from first-year physics.)

(d) If mass m is then perturbed slightly from this circular motion, what is the angular frequency Ω of the oscillations about the radius $r = r_0$? Express Ω in terms of ω, m, and M, where ω is your answer from part (c). [If your answer for part (d) makes no sense (e.g. if Ω appears to be imaginary), then you may have forgotten to make proper use of your answer for part (b).]

let $r = r_0 + \varepsilon \Rightarrow \ddot{\varepsilon} = \ddot{r} = \dfrac{mr\dot{\theta}^2 - Mg}{m+M} = \dfrac{mr\dot{\theta}^2}{m+M} - \dfrac{Mg}{m+M}$

$$\dot{\theta} = \frac{L}{mr^2} \Rightarrow \dot{\theta}^2 = \frac{L^2}{m^2 r^4}$$

$$\ddot{\varepsilon} = \left(\frac{mr}{m+M}\right)\left(\frac{L^2}{m^2r^4}\right) - \frac{Mg}{m+M} = \frac{L^2}{m(m+M)r^3} - \frac{Mg}{m+M} \equiv f(r)$$

$\ddot{\varepsilon} \simeq f(r_0) + \varepsilon f'(r_0)$ $\qquad f'(r_0) = \dfrac{-3L^2}{m(m+M)r_0^4}$

$f(r_0) \to 0$ by choice of r_0

$$\ddot{\varepsilon} = -\frac{3L^2}{m(m+M)r_0^4}\varepsilon = -\frac{3(mr_0^2\omega)^2}{m(m+M)r_0^4} = -\underbrace{\boxed{\frac{3m}{m+M}\omega^2}}_{\Omega^2}\varepsilon$$

$$\boxed{\Omega = \omega\sqrt{\frac{3m}{m+M}}}$$

note minus sign!

Where clearly Ω and ω have same units — good check to make a habit of.

Note: $\dot{\theta}$ is not constant when r varies, but L is constant. If you got $\ddot{\varepsilon} = +\Omega^2\varepsilon$ then you made that mistake.

Problem 2. (30%)
Two particles, each of mass m, are joined by a massless spring of natural (relaxed) length L and force constant k. Let z_1 denote the height above the table of particle 1, and let z_2 denote the height above the table of particle 2. Via some unspecified mechanism, the particles are constrained to move only along this one vertical axis, so there are no x or y coordinates to consider. Assume that $z_1 > z_2$.

(a) Write the Lagrangian for the two-particle system in terms of the total mass M, the reduced mass μ, the COM coordinate Z_{cm}, and the relative coordinate $z = z_1 - z_2$.

$$\mathcal{L} = \frac{1}{2} M \dot{Z}_{cm}^2 + \frac{1}{2}\mu \dot{z}^2 - M g Z_{cm} - \frac{1}{2} k (z - L)^2$$

where $M = 2m$, $\mu = m/2$, $Z_{cm} = \frac{1}{2}(z_1 + z_2)$

Oops, the initial conditions for this problem need a minor tweak for the problem to make complete sense. (There is a Taylor erratum for this.) Other than that, I still think it is a pretty good problem, in that it nicely illustrates the simplification that comes from separating the motion into CM motion and relative motion.

(b) Write the Lagrange equation of motion for Z_{cm} and write the Lagrange equation of motion for the relative coordinate z.

$$M \ddot{Z}_{cm} = -Mg \Rightarrow \ddot{Z}_{cm} = -g$$

(makes sense from first-year physics)

$$\mu \ddot{z} = -k(z - L)$$

$$\ddot{z} = -\frac{k}{\mu}(z - L)$$

(oscillatory motion: seems plausible from first-year physics, with equilibrium length L and reduced mass μ.)

$\boxed{\text{A physics major should, through practice, recognize these two differential equations and know how to solve them by inspection + verification.}}$

Initially particle 2 is resting on a table and I am holding particle 1 vertically above particle 2 at a height L (which equals, as noted above, the relaxed length of the spring). At time $t = 0$, I project particle 1 vertically upward with initial velocity v_0.

(c) Find the positions, $z_1(t)$ and $z_2(t)$, of the two particles at any subsequent time t, before either particle returns to the table. Assume that v_0 is small enough that the two particles never collide. Since the initial conditions have been fully specified, there should be no undetermined constants in your answers. That is, you should express $z_1(t)$ and $z_2(t)$ entirely in terms of t and the given constants v_0, L, g, k, and m. To simplify your expressions, you can — if you wish — define a new constant ω_0 in terms of other given constants.

$$z_{cm}(0) = \frac{L}{2} \qquad \dot{z}_{cm}(0) = \frac{v_0}{2} \qquad \omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{2k}{m}}$$

$$z_{cm}(t) = \frac{L}{2} + \frac{v_0 t}{2} - \frac{1}{2} g t^2$$

(familiar from kinematics)

$$z(0) = L \qquad \dot{z}(0) = v_0 \qquad \text{note } \ddot{z}(0) = 0 \text{ (equilibrium)}$$

$$z(t) = L + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$\boxed{\text{check:}}$ $\dot{z} = v_0 \cos(\omega_0 t)$

$\ddot{z} = -\omega_0 v_0 \sin(\omega_0 t)$

$-\frac{k}{\mu}(z - L) = -\omega_0 v_0 \sin(\omega_0 t) = \ddot{z}$ ✓

$$z_1(t) = z_{cm}(t) + \frac{1}{2} z(t) \quad \left(\text{since } m_1 = m_2 = \frac{M}{2}\right)$$

$$z_2(t) = z_{cm}(t) - \frac{1}{2} z(t)$$

$$z_1 = L + \frac{v_0 t}{2} - \frac{1}{2} g t^2 + \frac{v_0}{2\omega_0} \sin(\omega_0 t)$$

$$z_2 = 0 + \frac{v_0 t}{2} - \frac{1}{2} g t^2 - \frac{v_0}{2\omega_0} \sin(\omega_0 t)$$

(check: notice that all terms have same dimensions: distance)

check: $z_1(0) = L \qquad \dot{z}_1(0) = v_0$

$z_2(0) = 0 \qquad \dot{z}_2(0) = 0$

Problem 3. (35%)
At a polar angle θ (colatitude), a projectile is launched directly upward with initial speed v_0.

(a) Working to first order in Earth's rotational velocity Ω, calculate the eastward deflection, $x(t)$, due to the Coriolis force as a function of the time t since the projectile was launched. [Assume that air resistance is negligible and that $\boldsymbol{g}$ is a constant throughout the flight. I recommend first neglecting the Coriolis force and writing the "zeroth order" $z(t)$ (upward) for ordinary projectile motion; then use this zeroth-order trajectory to calculate the first-order Coriolis deflection, after evaluating $\boldsymbol{\Omega}$ in terms of the local $\hat{\boldsymbol{x}}$ (east), $\hat{\boldsymbol{y}}$ (north), $\hat{\boldsymbol{z}}$ (up) axes.] If you happen to know the right answer, you **cannot** just write it down. You **must** show how to work it out step by step. You should get a t^2 term and a t^3 term, each of whose coefficients should be first-order in Ω.

0th order: $z(t) = v_0 t - \frac{1}{2} g t^2 \qquad v_z(t) = v_0 - gt$

$\underset{\sim}{\Omega} = \hat{z}\,\Omega\cos\theta + \hat{y}\,\Omega\sin\theta \qquad \underset{\sim}{F}_{cor} = 2m\,\underset{\sim}{v} \times \underset{\sim}{\Omega}$

$\ddot{x} = 2(\underset{\sim}{v} \times \underset{\sim}{\Omega})_x = 2v_y\Omega_z - 2v_z\Omega_y = -2(v_0 - gt)\Omega\sin\theta$

$\ddot{x} = -2v_0\Omega\sin\theta + 2g(\Omega\sin\theta)t$

$\dot{x} = -2(v_0\Omega\sin\theta)t + (g\Omega\sin\theta)t^2$

$x(t) = -(v_0\Omega\sin\theta)t^2 + \frac{1}{3}(g\Omega\sin\theta)t^3$

$$z(t) = V_0 t - \frac{1}{2} g t^2$$

(b) Plugging in the zeroth-order time-of-flight (at which the projectile hits the ground), calculate the Coriolis deflection x_f, which should be proportional to Ω (and should have dimensions of distance).

$$z=0 \Rightarrow V_0 = \frac{1}{2} g t_f \Rightarrow t_f = \frac{2V_0}{g}$$

$$X_f = -(V_0 \Omega \sin\theta)\left(\frac{2V_0}{g}\right)^2 + \frac{1}{3}(g\Omega \sin\theta)\left(\frac{2V_0}{g}\right)^3$$

$$X_f = \left(-4 + \frac{8}{3}\right)\frac{V_0^3 \Omega \sin\theta}{g^2} = \boxed{-\frac{4}{3}\frac{V_0^3 \Omega \sin\theta}{g^2} = X_f}$$

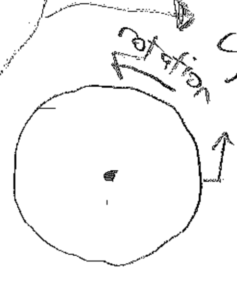
$$\frac{m^3}{s^3}\ \frac{1}{s}\ \frac{s^2}{m}\ \frac{s^2}{m} = \text{meters} \checkmark$$

(c) Is the Coriolis deflection that you calculated in (b) in fact eastward, or does it turn out to be westward? Offer an intuitive explanation for why the sign of your answer makes sense for an object tossed straight upward from the ground.

Remarkably, the net deflection is westward, vs. our known eastward deflection for an object dropped straight down. The westward deflection during the upward portion must exceed the eastward deflection during downward portion.

If you said this you got +1 bonus

Seen from inertial frame, initial tangential speed is $\Omega R \sin\theta$. As r increases, the tangential speed needed to keep up with Earth's rotation would be $\Omega(R\sin\theta + z)$. So the object falls behind as Earth rotates eastward. Object lands farther west than launch.



viewed from above north pole