Physics 351 — Wednesday, January 10, 2018

- Chapers 1–5 mostly review "freshman physics," so we'll go through them very quickly in the first few days of class.
- Read Chapters 1+2 for Friday.
- Read Chapter 3 (momentum & angular momentum) during the long weekend. I also very strongly encourage you to install Mathematica & do the Mathematica "hands-on start" video.
- Skim Chapter 4 (energy) for next Wednesday (1/17).
- Read Chapter 5 (oscillations) for next Friday (1/19).
- Homework #1 due on Friday 1/26. I'll hand it out next week.
- See also syllabus handout: read it at home instead of listening to me read it to you here!
- Remember online questions for each reading assignment.
- I need to remember to discuss Perusall for textbook.
- Today: course overview.

What is “analytical mechanics?”

- Course catalog: “An intermediate course in the statics and dynamics of particles and rigid bodies. Lagrangian dynamics, central forces, non-inertial reference frames, and rigid bodies.”
- We’ll review and extend the mechanics that you studied in first-year physics.
- The course re-visits mechanics in a way that takes advantage of some of the math you’ve learned since you started college. Brush up on familiar physics to gain a deeper understanding.
- Most importantly, “analytical mechanics” will give you an extended toolkit for systematically taking on more complicated mechanics problems.
- In particular, the Lagrangian formalism provides a systematic approach that simplifies many tricky mechanics problems.
- The Lagrangian and (closely related) Hamiltonian approaches also form a bridge between classical mechanics and quantum mechanics. So I think taking 351 early can be helpful for QM.

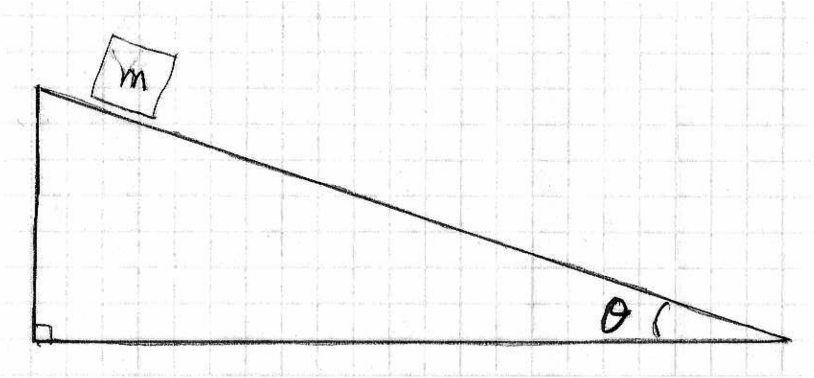
What is "analytical mechanics?"

- Let's look briefly at a *really simple* first-year physics problem.
- Then let's look at a trickier first-year physics problem.
- Then we'll see how much more quickly we can write down the answer to the trickier problem, once we've studied Lagrangian mechanics (later this month, formally; and sooner, casually).
  - This is just to give you some hint of where we're headed.

By the way, most days I'll use the projector and the blackboard side-by-side. I'll try to use slides to display information (which would be boring to watch me write out) and the board to go step-by-step (where following the reasoning behind each step is important). As I get to know you, I'll try to adapt to what works best for you.

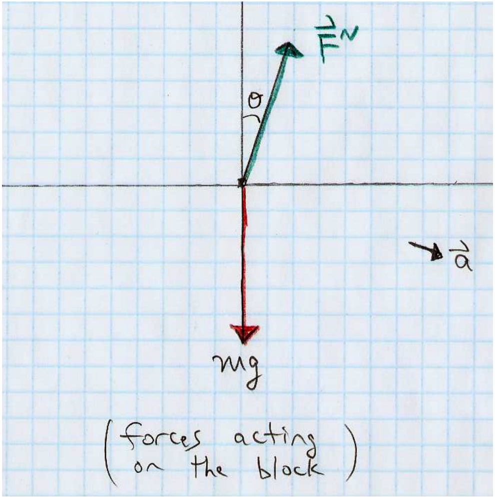
The slides will be online afterward, so taking notes is optional. But I'll often ask you to spend a moment calculating with your neighbor, before I work through the same problem on the board.

Example of what is **not** "analytical mechanics:" Simple freshman physics problem: a block of mass m slides down a **frictionless** ramp. Find the equation of motion of the block.



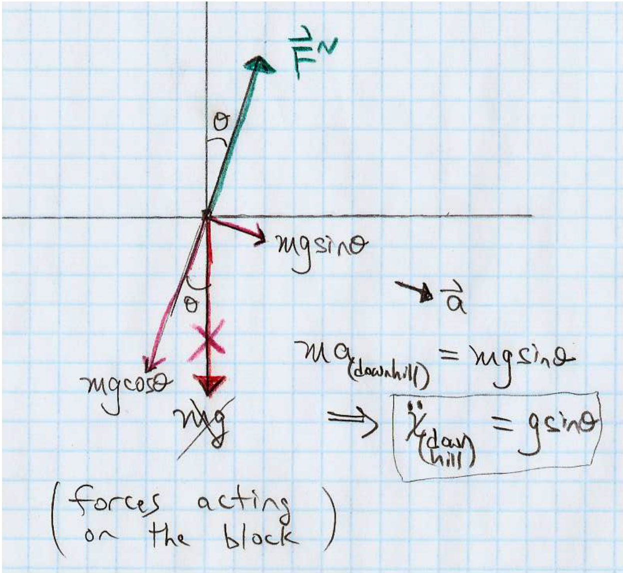
You write down all forces acting ON the block. You impose relevant constraints (i.e. the forces perpendicular to the wedge sum to zero). Then you use $\sum \vec{F} = m\vec{a}$ to write down the equation of motion.

Write down (or draw) the forces acting on the block: (1) gravity points downward; (2) the "normal force" points perpendicular to the surface. In most freshman physics problems, we'd include friction as well, but let's consider the frictionless case.

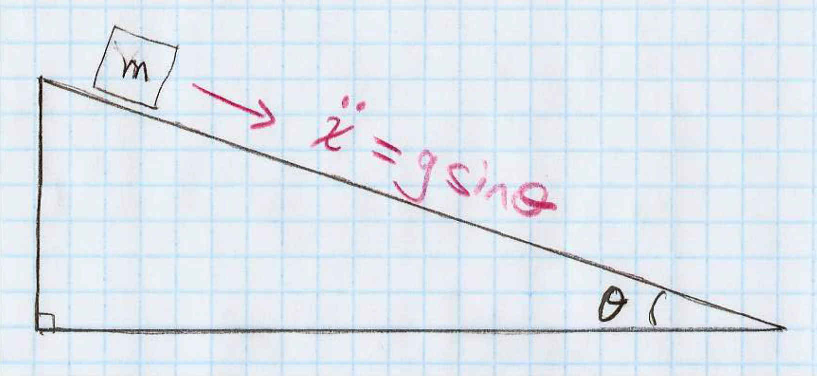


In this case, it's obvious which way the acceleration $\vec{a}$ will point.

Since we know the motion will be parallel to the surface, it makes sense to "decompose" gravity's $m\vec{g}$ into "normal" and "downhill" components. I never once drew one of these diagrams until the first time I taught a Physics-101-like course (Phys 008). Have you?

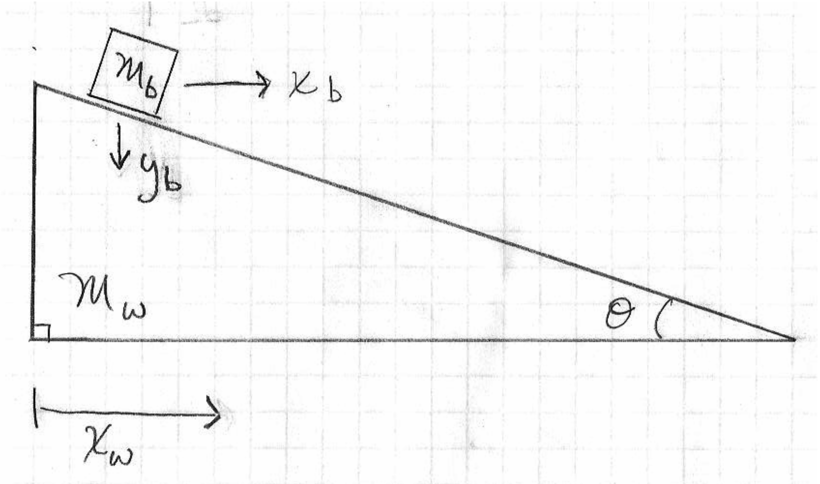


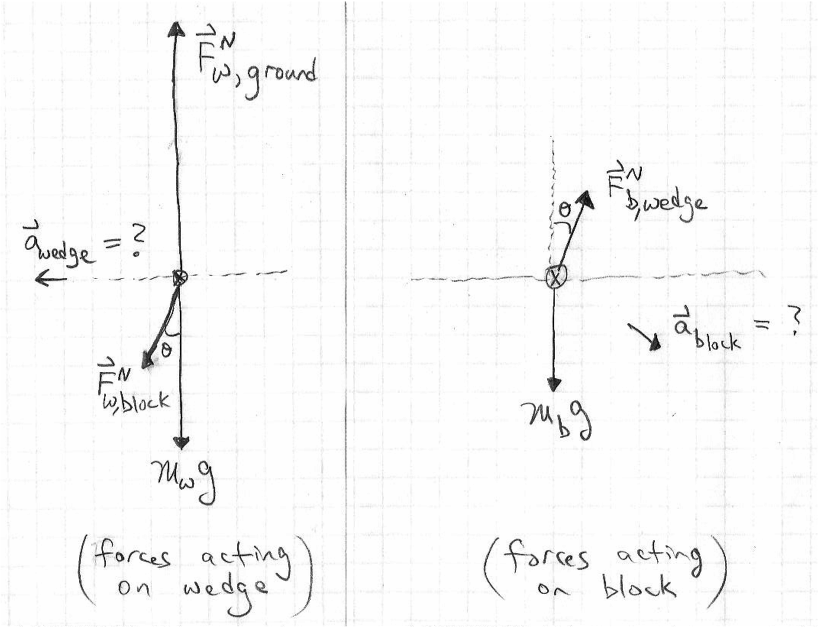
If you take "x" to point in the downhill direction, then Newton's second law gives you $\ddot{x} = \frac{1}{m}\sum F_x = g\sin\theta$.



This is not what "analytical mechanics" is about!

Here is a trickier freshman physics problem, which was a favorite of my first physics teacher, Mr. Rodriguez. Let's first solve it using familiar methods. Block of mass m_b slides down face of frictionless wedge of mass m_w. The wedge sits on a frictionless horizontal table. Find the EOM for wedge and block. (Measure x_b in Earth frame.) Here, a force diagram may be helpful — draw on board.





EOM (on board): x_w, x_b, y_b; then constraint.

Measure y_b downward, x_b & x_w to the right.
Measure x_b, y_b in the earth reference frame.
Use Newton's second law to write equations of motion:

(1) $m_w \ddot{x}_w = -F^N \sin\theta \Rightarrow F^N = -m_w \ddot{x}_w / \sin\theta$

(2) $m_b \ddot{x}_b = F^N \sin\theta$

(3) $m_b \ddot{y}_b = m_b g - F^N \cos\theta$

constraint equation: block remains on wedge

(4) $\tan\theta = \dfrac{\Delta y_b}{\Delta(x_b - x_w)} = \dfrac{\ddot{y}_b}{\ddot{x}_b - \ddot{x}_w} \Rightarrow \ddot{y}_b = (\ddot{x}_b - \ddot{x}_w)\tan\theta$

BTW, notice that $m_b \ddot{x}_b + m_w \ddot{x}_w = 0$. Momentum conservation implies that C.o.M.'s horizontal velocity is constant. $(=0)$

Shall I do the algebra on the board, or just show you the answer?
First convince yourself that this is somewhat tedious!

$$③ + ④ \Rightarrow m_b(\ddot{x}_b - \ddot{x}_w)\tan\theta = m_b g - F^N\cos\theta$$

$$② \Rightarrow F^N\sin\theta\tan\theta - m_b\ddot{x}_w\tan\theta = m_b g - F^N\cos\theta$$

$$\Rightarrow F^N(\sin\theta\tan\theta + \cos\theta) - m_b\ddot{x}_w\tan\theta = m_b g$$

$$① \Rightarrow -\frac{m_w\ddot{x}_w}{\sin\theta}(\sin\theta\tan\theta + \cos\theta) - m_b\ddot{x}_w\tan\theta = m_b g$$

$$\Rightarrow m_w\ddot{x}_w\left(\tan\theta + \frac{\cos\theta}{\sin\theta}\right) + m_b\ddot{x}_w\tan\theta = -m_b g$$

$$\ddot{x}_w\left[m_w\sin^2\theta + m_w\cos^2\theta + m_b\sin^2\theta\right] = -m_b g\sin\theta\cos\theta$$

$$\ddot{x}_w\left[m_w + m_b\sin^2\theta\right] = -m_b g\sin\theta\cos\theta$$

$$\Rightarrow \ddot{x}_w = -\frac{m_b g\sin\theta\cos\theta}{m_w + m_b\sin^2\theta}$$

Digression: We could just ask Mathematica to do the messy algebra for us:

In[56]:=
```
ClearAll["Global`*"];
solution = Reduce[ {
    mw xwdotdot == -fNormal Sin[θ],
    mb xbdotdot == fNormal Sin[θ],
    mb ybdotdot == mb g - fNormal Cos[θ],
    ybdotdot == (xbdotdot - xwdotdot) Tan[θ],
    Sin[θ] ≠ 0, Cos[θ] ≠ 0, mw ≠ 0, mb ≠ 0, g ≠ 0
  }, {xwdotdot, xbdotdot, ybdotdot, fNormal}]
```

Out[57]=

$$\mathrm{mw}\,\mathrm{Cos}[\theta] + \mathrm{mb}\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta] + \mathrm{mw}\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta] \neq 0 \;\&\&$$

$$\mathrm{xwdotdot} == -\frac{\mathrm{g}\,\mathrm{mb}\,\mathrm{Sin}[\theta]}{\mathrm{mw}\,\mathrm{Cos}[\theta] + \mathrm{mb}\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta] + \mathrm{mw}\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta]} \;\&\&$$

$$\mathrm{mb} \neq 0 \;\&\&\; \mathrm{xbdotdot} == -\frac{\mathrm{mw}\,\mathrm{xwdotdot}}{\mathrm{mb}} \;\&\&$$

$$\mathrm{ybdotdot} == (\mathrm{xbdotdot} - \mathrm{xwdotdot})\,\mathrm{Tan}[\theta] \;\&\&\; \mathrm{Sin}[\theta] \neq 0 \;\&\&$$

$$\mathrm{fNormal} == -\mathrm{mw}\,\mathrm{xwdotdot}\,\mathrm{Csc}[\theta] \;\&\&\; \mathrm{g}\,\mathrm{mw}\,\mathrm{Cos}[\theta] \neq 0$$

The second expression (the $\ddot{x}_w$ result) is the one we want.

In[58]:= `solution[[2]]`

Out[58]= $$\text{xwdotdot} == -\frac{g\,mb\,\text{Sin}[\theta]}{mw\,\text{Cos}[\theta] + mb\,\text{Sin}[\theta]\,\text{Tan}[\theta] + mw\,\text{Sin}[\theta]\,\text{Tan}[\theta]}$$

In[59]:= `TrigExpand[solution[[2]]]`

Out[59]= $$\text{xwdotdot} == \frac{2\,g\,mb\,\text{Cos}[\theta]\,\text{Sin}[\theta]}{-mb - 2\,mw + mb\,\text{Cos}[\theta]^2 - mb\,\text{Sin}[\theta]^2}$$

If you rewrite Mathematica's result using $\cos^2\theta = 1 - \sin^2\theta$, you get the result we got. You can actually tell Mathematica to do that replacement for you:

In[83]:= `solution[[2]]`

Out[83]= $$\text{xwdotdot} == -\frac{g\,mb\,\text{Sin}[\theta]}{mw\,\text{Cos}[\theta] + mb\,\text{Sin}[\theta]\,\text{Tan}[\theta] + mw\,\text{Sin}[\theta]\,\text{Tan}[\theta]}$$

In[84]:= `TrigExpand[solution[[2]]]`

Out[84]= $$\text{xwdotdot} == \frac{2\,g\,mb\,\text{Cos}[\theta]\,\text{Sin}[\theta]}{-mb - 2\,mw + mb\,\text{Cos}[\theta]^2 - mb\,\text{Sin}[\theta]^2}$$

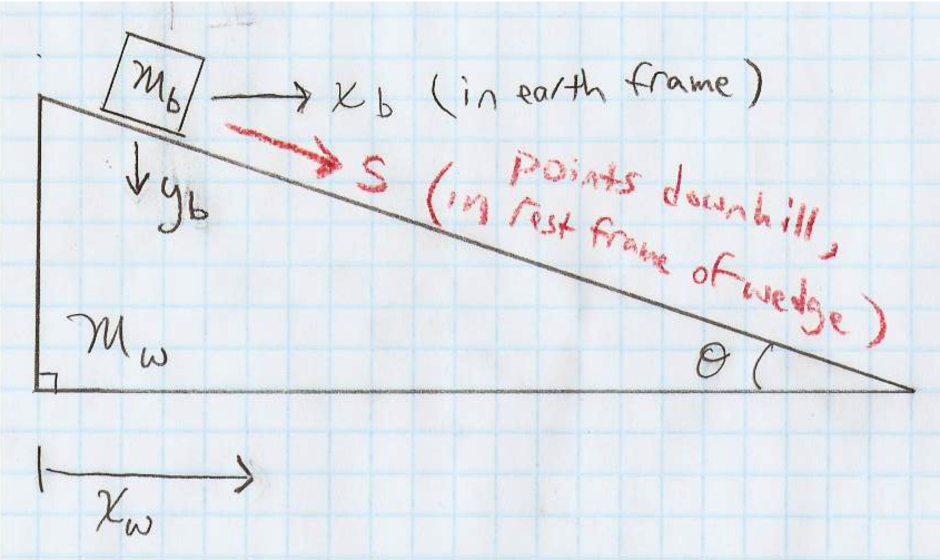
In[85]:= `% /. {Cos[θ]^2 → 1 - Sin[θ]^2}`

Out[85]= $$\text{xwdotdot} == \frac{2\,g\,mb\,\text{Cos}[\theta]\,\text{Sin}[\theta]}{-mb - 2\,mw - mb\,\text{Sin}[\theta]^2 + mb\,\left(1 - \text{Sin}[\theta]^2\right)}$$

In[86]:= `Simplify[%]`

Out[86]= $$\text{xwdotdot} == -\frac{g\,mb\,\text{Cos}[\theta]\,\text{Sin}[\theta]}{mw + mb\,\text{Sin}[\theta]^2}$$

Mathematica is quite powerful, but it takes a while to learn how to tell it just what you want it to do. (End of digression.)



Now that we have $\ddot{x}_w$ we can work out what I'm calling $\ddot{s}$ (downhill acceleration, in rest frame of wedge), to compare with the much simpler $m_w = \infty$ case. (On board, if time allows.)

$$\ddot{x}_w = -\frac{m_b g \sin\theta\cos\theta}{m_w + m_b\sin^2\theta}$$

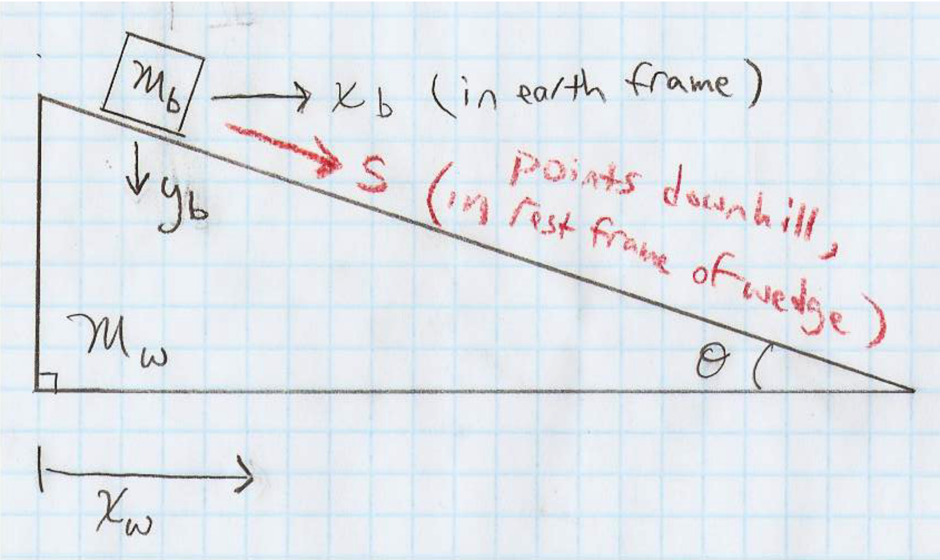
$$\ddot{x}_b = -\frac{m_w}{m_b}\ddot{x}_w = \frac{m_w g \sin\theta\cos\theta}{m_w + m_b\sin^2\theta}$$

$$\ddot{x}_b - \ddot{x}_w = \frac{(m_w + m_b) g \sin\theta\cos\theta}{m_w + m_b\sin^2\theta}$$

$$\ddot{s} = \frac{\ddot{x}_b - \ddot{x}_w}{\cos\theta} = (g\sin\theta)\frac{m_w + m_b}{m_w + m_b\sin^2\theta}$$

acceleration of block down hill, in rest frame of wedge

$$\ddot{s} = \frac{g\sin\theta}{1 - \frac{m_b\cos^2\theta}{m_w + m_b}}$$



The Lagrangian approach (Chapter 7) starts by simply writing down the "Lagrangian" $\mathcal{L} = T - U$ in terms of s and x_w. So we bypass forces and just write down energies. (Let's try it.)

$$T = \frac{1}{2} m_b (\dot{x}_b^2 + \dot{y}_b^2) + \frac{1}{2} m_w \dot{x}_w^2$$

$$U = -m_b g\, y_b = -m_b g s \sin\theta$$

$$y_b = s \sin\theta$$

$$x_b = x_w + s\cos\theta$$

$$T = \frac{1}{2} m_b \left[(\dot{x}_w + \dot{s}\cos\theta)^2 + (\dot{s}\sin\theta)^2\right] + \frac{1}{2} m_w \dot{x}_w^2$$

$$T = \frac{m_b}{2}\left[\dot{x}_w^2 + 2\dot{x}_w\dot{s}\cos\theta + \dot{s}^2\right] + \frac{m_w}{2}\dot{x}_w^2$$

$$L = T - U = \frac{m_b}{2}\left[\dot{x}_w^2 + 2\dot{x}_w\dot{s}\cos\theta + \dot{s}^2\right] + \frac{m_w}{2}\dot{x}_w^2 + m_b g s \sin\theta$$

So the "Lagrangian" ($\mathcal{L} = T - U$) for this system is

$$\mathcal{L} = T - U = \frac{1}{2}(m_b + m_w)\dot{x}_w^2 + \frac{1}{2} m_b(\dot{s}^2 + 2\dot{s}\dot{x}_w\cos\theta) + m_b g s\sin\theta$$

The recipe we'll learn (and derive!) in Chapters 6+7 is:

- First write down $\mathcal{L} = T - U$, which is generally much easier than writing down the corresponding forces, because it often requires much less thinking to write down the energies than to figure out the forces.

$$\mathcal{L} = T - U = \frac{1}{2}(m_b + m_w)\dot{x}_w^2 + \frac{1}{2}m_b(\dot{s}^2 + 2\dot{s}\dot{x}_w \cos\theta) + m_b g s \sin\theta$$

- Write down an EOM for each coordinate (s and x_w in this case), using this "magic incanation" (which we'll later derive).

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{s}} \qquad\qquad \frac{\partial \mathcal{L}}{\partial x_w} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_w}$$

- This should seem completely mysterious to you at this point. I just want to give you a preview of things to come.

Let's try it anyway!

$$\frac{\partial L}{\partial s} = m_b g \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left[\left(\frac{m_b}{2}\right)\left(2\dot{s} + 2\dot{x}_\omega \cos\theta\right)\right] = m_b\left(\ddot{s} + \ddot{x}_\omega \cos\theta\right)$$

$$\Rightarrow \quad \ddot{s} + \ddot{x}_\omega \cos\theta = g\sin\theta$$

$$\frac{\partial L}{\partial x_\omega} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\omega}\right) = \frac{d}{dt}\left[\frac{1}{2}(m_b+m_\omega)(2\dot{x}_\omega) + \frac{m_b}{2}\left(2\dot{s}\cos\theta\right)\right] = (m_b+m_\omega)\ddot{x}_\omega + m_b \ddot{s}\cos\theta$$

$$\Rightarrow \quad (m_b+m_\omega)\,\ddot{x}_\omega + (m_b\cos\theta)\,\ddot{s} = 0$$

$$\Rightarrow \quad \ddot{x}_\omega = -\left(\frac{m_b\cos\theta}{m_b+m_\omega}\right)\ddot{s}$$

$$\ddot{s}\left[1 - \frac{m_b\cos^2\theta}{m_b+m_\omega}\right] = g\sin\theta$$

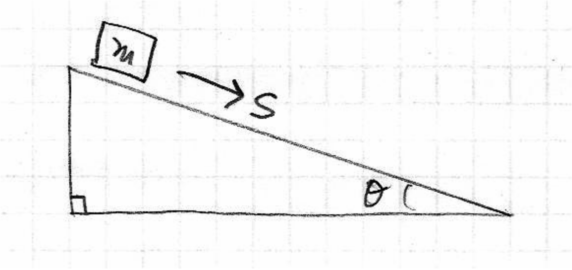
$$\Rightarrow \quad \boxed{\ddot{s} = \frac{g\sin\theta}{1 - \frac{m_b\cos^2\theta}{m_b+m_\omega}}}$$

So the Lagrangian formalism allows us to find the exact same EOM as we get using familiar Newtonian methods, but ...

- Requiring roughly half as much tedious algebra.
- Allowing us to bypass completely any mention of the "normal force" that constrains the block to stay on the wedge.
- Without requiring us to think too hard about the fact that coordinate s is measured w.r.t. an accelerating (non-inertial) frame of reference. We just used the definition of s to **write the energies w.r.t. an inertial frame.**

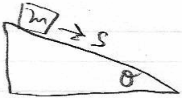
If you're not yet convinced, note that the simplifications, w.r.t. the force method, are far greater when working in polar coordinates. (You'll see in tomorrow night's reading that Newton's 2nd law looks a bit nasty in polar coordinates.)

Do you want to try the "trivial" version of the freshman-physics problem with the Lagrangian formalism?



1. Choose coordinate: s points downhill.
2. Write down T in terms of s and $\dot{s}$.
3. Write down U in terms of s and $\dot{s}$.
4. Write down $\mathcal{L} = T - U$.
5. Equation of motion is given by

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{s}}$$



$$T = \tfrac{1}{2} m \dot{s}^2 \qquad U = -mgs\sin\theta$$

$$L = T - U = \tfrac{1}{2} m \dot{s}^2 + mgs\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{\partial L}{\partial s} \Rightarrow \frac{d}{dt}\left(m\dot{s}\right) = mg\sin\theta$$

$$m\ddot{s} = mg\sin\theta$$

$$\boxed{\ddot{s} = g\sin\theta} \quad \text{expected}$$

The "magic incantation" is called the Euler-Lagrange equation. We'll meet it officially in Chapters 6 and 7.

I generally **will not** spend class repeating the contents of the book. Instead, I'll assume you've done the **required reading**, so we can focus class time on assimilating the key ideas and working through example problems together. The 2015 and 2017 students generally agreed that this approach was a good fit for them:

- "I learned so much from this course, from the textbook, from the homework problems, from the strategy of teaching. The homework assignments require a lot of work and time, as does the reading, but you are rewarded by actually understanding the material."
- "I liked the approach of solving problems during class rather than just repeating the book. I did like Bill's emphasis on reading the textbook by Taylor, which is quite exemplary, and I think this focus was very helpful to my understanding."

But it only works if you actually read the textbook before class! So let me show you how you collect your credit for doing the reading.

http://positron.hep.upenn.edu/q351

positron.hep.upenn.edu/q351

Physics 351 reading assignments page

Signed in as Bill Ashmanskas

sign out | my info

Monday	Wednesday	Friday
		Jan 12 — *Chapters 1+2*
Jan 15 — *Chapter 3*	Jan 17 — *Chapter 4*	Jan 19 — *Chapter 5*
Jan 22 — *Chapter 6*	Jan 24 — *more Mathematica*	Jan 26 — *Homework 1*
Jan 29 — *Chapter 7a*		Feb 2 — *Homework 2*
Feb 5 — *Chapter 7b*		Feb 9 — *Homework 3*
Feb 12 — *Chapter 8*		Feb 16 — *Homework 4*
Feb 19 — *Chapter 9*		Feb 23 — *Homework 5*
Feb 26 — *Chapter 10a*		Mar 2 — *Homework 6*
Mar 12 — *Chapter 10b*		Mar 16 — *Homework 7*
		Mar 23 — *Homework 8*
		Mar 30 — *Homework 9*
Apr 2 — *Chapter 11*		Apr 6 — *Homework 10*
Apr 9 — *Chapter 13*	Apr 11 — *Morin's Chapter 15*	Apr 13 — *Homework 11*
Apr 16 — *Chapter 12*	Apr 18 — *optional/XC: Chapter 14*	Apr 20 — *Homework 12*
Apr 23 — *Feynman/Hibbs*	Apr 25 — *Feynman Lectures II.40 and II.41*	Apr 27 — *optional/XC: Chapter 16*

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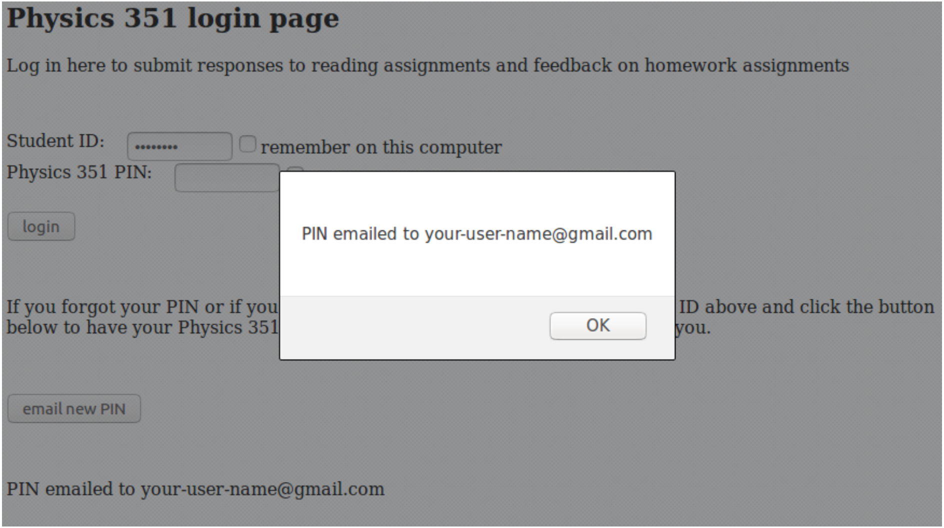

Physics 351 PIN

Inbox x

Bill Ashmanskas

to me

Hello Bill Ashmanskas,

Your Physics 351 PIN is 1234

I also need to show you how (if you wish) to use Perusall to read the textbook online, which lets you share questions and annotations and respond to your classmates' questions and annotations. If you think that doing so will enhance your learning, then you can learn extra credit for reading that way. The choice is yours.

Let's look briefly through the list of topics we'll cover this term, and then the course policies, etc. You have a printed copy in your hand. (We may put this off until Friday if time is short.)

http://positron.hep.upenn.edu/p351#schedule

http://positron.hep.upenn.edu/p351

From Richard Feynman (*Feynman Lectures*):

- "I think, however that there isn't any solution to this problem of education other than to realize that the best teaching can be done only when there is a direct individual relationship between a student and a good teacher — a situation in which the student discusses the ideas, thinks about the things, and talks about the things. It's impossible to learn very much by simply sitting in a lecture . . . ."

From Mary Boas (*Mathematical Methods in the Physical Sciences*):

- "One point about your study of this material cannot be emphasized too strongly: To use mathematics effectively in applications, you need not just knowledge but *skill*. Skill can be obtained only through practice. You can obtain a certain superficial *knowledge* of mathematics by listening to lectures, but you cannot obtain *skill* this way. How many students have I heard say, 'It looks so easy when you do it,' or 'I understand it but I can't do the problems!' Such statements show lack of practice . . . ."

To whatever degree is feasible, I'll try to focus the time you spend for this course on solving (worthwhile) problems. I'll also try to facilitate your "talking about the things" with me, with Grace, and with each other as much as possible.

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- Did we remember to discuss Perusall?