Physics 351 — Wednesday, January 17, 2018

One great feature of Taylor's book is the gentle math review presented alongside the physics. One worthwhile math trick from Ch2 is separation of varibles. (The many detailed drag-force results from Ch2 are not worth remembering, but the math methods he illustrates are valuable.) Let's try one separation-of-variables problem together.

A mass m has initial velocity v_0 at $t = 0$ and coasts along the x axis with drag force $F(v) = -cv^3$. Find $v(t)$.

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$$F(v) = -cv^3$$

$$m\ddot{x} = -c(\dot{x})^3 \rightarrow m\dot{v} = -cv^3 \rightarrow \frac{\dot{v}}{v^3} = -\frac{c}{m}$$

$$\int_{v'=v_0}^{v'=v} \frac{dv'}{(v')^3} = -\frac{c}{m}\int_{t'=0}^{t} dt' \rightarrow \left[\frac{-1}{2(v')^2}\right]_{v_0}^{v} = -\frac{c}{m}(t-0)$$

$$\frac{1}{v^2} - \frac{1}{v_0^2} = \frac{2ct}{m} \rightarrow \frac{1}{v^2} = \left(\frac{2ct}{m} + \frac{1}{v_0^2}\right)$$

$$v(t) = \frac{1}{\sqrt{\frac{2ct}{m} + \frac{1}{v_0^2}}} = \frac{\sqrt{\frac{m}{2c}}}{\sqrt{t + \frac{m}{2cv_0^2}}}$$

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Incidentally, here's one way to solve the same problem using Mathematica. As you learn more and more of Mathematica's obscure syntax, you can solve problems with less and less typing.

```
In[1]:= solution = DSolve[{v'[t] / v[t]^3 == -c / m, v[0] == v0}, v, t]
```

Out[1]= $\left\{\left\{v \to \text{Function}\left[\{t\}, -\frac{\sqrt{m}}{\sqrt{\frac{m+2\,c\,t\,v0^2}{v0^2}}}\right]\right\}, \left\{v \to \text{Function}\left[\{t\}, \frac{\sqrt{m}}{\sqrt{\frac{m+2\,c\,t\,v0^2}{v0^2}}}\right]\right\}\right\}$

```
In[2]:= (v /. solution[[2]])[t]
```

Out[2]= $\frac{\sqrt{m}}{\sqrt{\frac{m+2\,c\,t\,v0^2}{v0^2}}}$

```
In[3]:= FullSimplify[Out[2]]
```

Out[3]= $\frac{\sqrt{m}}{\sqrt{2\,c\,t + \frac{m}{v0^2}}}$

Obscure: The mysterious `/.` stands for the `ReplaceAll[]` command, which can "plug in" solutions or values to an expression.

```
In[236]:= a + 2 b + 10

Out[236]= 10 + a + 2 b

In[237]:= a + 2 b + 10 /. {a → 1}

Out[237]= 11 + 2 b

In[238]:= a + 2 b + 10 /. {a → 1, b → 2}

Out[238]= 15
```

The main value of chapter 2 (aside from seeing how air drag can be handled, which is fun but not something we will build upon in the course) is Taylor's showing you some useful mathematical methods. Since many of you have not yet had much occasion to use complex exponentials in physics courses, let's try this problem together:

A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, $\mathbf{E}$ pointing in the y direction and $\mathbf{B}$ in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{x0}$ (positive or negative).

(a) Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

$$m\dot{v}_y = qE_y + q(\vec{v}\times\vec{B})_y \qquad \text{etc.}$$

$$\vec{E} = E\hat{y} \qquad \vec{B} = B\hat{z}$$

$$(\vec{v}\times\vec{B})_x = v_y B_z - v_z B_y = B v_y$$

$$(\vec{v}\times\vec{B})_y = v_z B_x - v_x B_z = -B v_x$$

$$(\vec{v}\times\vec{B})_z = v_x B_y - v_y B_x = 0$$

$$m\dot{v}_x = qBv_y$$
$$m\dot{v}_y = qE - qBv_x$$
$$m\dot{v}_z = 0$$

So the motion stays in the $z = 0$ plane since v_z is initially zero.

(b) Notice that if $v_{x0} = E/B$ (called the "drift speed"), then the particle moves undeflected through the fields. This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam with many different speeds.

(c) Now make the substitution $u_x = v_x - E/B$ and $u_y = v_y$. Then write $\dot{u}_x$ and $\dot{u}_y$ in terms of u_x and u_y.

let $u_x = v_x - \frac{E}{B}$ $\qquad u_y = v_y$

$$\dot{u}_x = \dot{v}_x = \frac{qB}{m} v_y = \boxed{\frac{qB}{m} u_y = \dot{u}_x}$$

$$\dot{u}_y = \dot{v}_y = \frac{q}{m} E - \frac{qB}{m}\left(u_x + \frac{E}{B}\right) = \boxed{-\frac{qB}{m} u_x = \dot{u}_y}$$

(d) Now let $\eta = u_x + iu_y$. Write $\dot{\eta}$ in terms of η. Try a solution $\eta(t) = Ae^{-i\omega t}$.

let $\eta = u_x + i u_y$

$$\dot{\eta} = \dot{u}_x + i\dot{u}_y = \frac{qB}{m} u_y - i \frac{qB}{m} u_x$$

$$= -\frac{iqB}{m}\left(u_x + i u_y\right) = -\frac{iqB}{m}\eta$$

$$\dot{\eta} = -i\omega\eta \quad \Rightarrow \quad \eta(t) = A e^{-i\omega t}$$

(d) Now un-substitute to get back $v_x(t)$ and $v_y(t)$. Then integrate (using $x(0) = y(0) = 0$) to find $x(t)$ and $y(t)$. Use the constants $\omega = qB/m$, $v_{\mathrm{drift}} = E/B$, $R = (v_{x0} - v_{\mathrm{drift}})/\omega$ to make the answer look nice.

$$\underset{\sim}{E} = E\hat{y} \qquad \underset{\sim}{B} = B\hat{z} \qquad \omega = \frac{qB}{m}$$

$$u_x + i u_y = \eta = \left(V_{x0} - \frac{E}{B}\right) e^{-i\omega t}$$

$$u_x = \left(V_{x0} - \frac{E}{B}\right)\cos(\omega t) \qquad u_y = \left(\frac{E}{B} - V_{x0}\right)\sin(\omega t)$$

$$V_x = \frac{E}{B} + \left(V_{x0} - \frac{E}{B}\right)\cos(\omega t) \qquad V_y = \left(\frac{E}{B} - V_{x0}\right)\sin(\omega t)$$

$$x = x_0 + \frac{E}{B} t + \boxed{\frac{\left(V_{x0} - \frac{E}{B}\right)}{qB/m}} \sin(\omega t) \qquad y = y_0 + \boxed{\frac{\left(V_{x0} - \frac{E}{B}\right)}{qB/m}} \cos(\omega t)$$

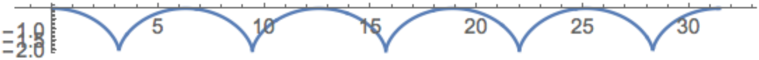
$$\equiv R \qquad\qquad \equiv R$$

$$x = \frac{E}{B} t + R\sin(\omega t) \qquad y = R\left(\cos(\omega t) - 1\right)$$

$$\text{if } x_0 = y_0 = 0$$

```
In[]:= ClearAll["Global`*"];
ω = 1.0;
R = 1.0;
vdrift = 1.0;
ParametricPlot[{vdrift t + R Sin[ω t], R (Cos[ω t] - 1)},
  {t, 0, 10 Pi}]
```

Out[]=




```
In[]:=
```

Chapter 3: I put this here just for your reference (not in class)

Center of mass (i = particle index here, not coordinate index):

$$\vec{R}_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i \, \vec{r}_i \qquad \Rightarrow \qquad \vec{v}_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i \, \vec{v}_i$$

Momentum of system is sum of particle momenta:

$$\vec{P}_{\rm sys} = \sum_i m_i \, \vec{v}_i = M_{\rm tot} \left[\frac{1}{M_{\rm tot}} \sum_i m_i \, \vec{v}_i \right] = M_{\rm tot} \, \vec{v}_{\rm cm}$$

So CM acceleration is (net external force) / (total mass)

$$\frac{{\rm d}\vec{P}_{\rm sys}}{{\rm d}t} = \sum \vec{F}_{\rm external} = M_{\rm tot} \, \vec{a}_{\rm cm}$$

(internal forces cancel out: 3rd-law "interaction pairs" sum to 0)

Classic example: rocket (of present mass $m(t)$) ejecting fuel

$$\frac{{\rm d}\vec{P}_{\rm sys}}{{\rm d}t} = m(t) \, \frac{{\rm d}\vec{v}_{\rm rocket}}{{\rm d}t} + \vec{v}_{\rm exhaust} \frac{{\rm d}m(t)}{{\rm d}t} = \vec{F}_{\rm external}$$

Angular momentum for a particle (depends on choice of origin!)

$$\vec{\ell} = \vec{r} \times \vec{p}$$

Define torque (a.k.a. "moment") $\vec{\tau}$ by analogy with $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$

$$\vec{\tau} \;\equiv\; \frac{\mathrm{d}\vec{\ell}}{\mathrm{d}t} \;=\; \vec{v} \times (m\vec{v}) \;+\; \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \;=\; \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \;=\; \vec{r} \times \vec{F}^{\mathrm{net}}$$

where last step (Newton's 2nd law) assumes inertial frame.

Angular momentum for a system of particles (i = particle index):

$$\vec{L}_{\mathrm{sys}} = \sum_i \vec{\ell}_i = \sum_i \vec{r}_i \times \vec{p}_i \qquad \Rightarrow \qquad \frac{\mathrm{d}\vec{L}_{\mathrm{sys}}}{\mathrm{d}t} = \sum_i \vec{r}_i \times \vec{F}_i^{\mathrm{net}}$$

Write out $\mathrm{d}\vec{L}_{\mathrm{sys}}/\mathrm{d}t$ term-by-term for two-particle case:

$$\frac{\mathrm{d}\vec{L}_{\mathrm{sys}}}{\mathrm{d}t} \;=\; \vec{r}_1 \times (\vec{F}_{1\mathrm{by}2} + \vec{F}_{1,\mathrm{ext}}) \;+\; \vec{r}_2 \times (\vec{F}_{2\mathrm{by}1} + \vec{F}_{2,\mathrm{ext}})$$

Write out $\mathrm{d}\vec{L}_{\rm sys}/\mathrm{d}t$ for two-particle case:

$$\frac{\mathrm{d}\vec{L}_{\rm sys}}{\mathrm{d}t} = \vec{r}_1 \times (\vec{F}_{12} + \vec{F}_{1,\rm ext}) \; + \; \vec{r}_2 \times (\vec{F}_{21} + \vec{F}_{2,\rm ext})$$

$$= \vec{r}_1 \times \vec{F}_{12} \; + \; \vec{r}_1 \times \vec{F}_{1,\rm ext} \; + \; \vec{r}_2 \times \vec{F}_{21} \; + \; \vec{r}_2 \times \vec{F}_{2,\rm ext}$$

now use Newton's third law:

$$= \vec{r}_1 \times \vec{F}_{12} \; + \; \vec{r}_1 \times \vec{F}_{1,\rm ext} \; + \; \vec{r}_2 \times (-\vec{F}_{12}) \; + \; \vec{r}_2 \times \vec{F}_{2,\rm ext}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} \; + \; \vec{r}_1 \times \vec{F}_{1,\rm ext} \; + \; \vec{r}_2 \times \vec{F}_{2,\rm ext}$$

now assume $\vec{F}_{12}$ points along $\vec{r}_1 - \vec{r}_2$ (is a "central" force):

$$\frac{\mathrm{d}\vec{L}_{\rm sys}}{\mathrm{d}t} = 0 \; + \; \vec{r}_1 \times \vec{F}_{1,\rm ext} \; + \; \vec{r}_2 \times \vec{F}_{2,\rm ext} \; = \; \vec{\tau}_{\rm external}$$

because $(\vec{r}_1 - \vec{r}_2) \times (\vec{r}_1 - \vec{r}_2) = 0$

So if there are no external torques, then $\vec{L}_{\rm sys}$ is constant.

(Tip: when proofs filled with $\sum$ symbols are hard to follow, try writing out the $N = 2$ or $N = 3$ case term-by-term.)

As long as all internal forces are central and obey Third Law (which is a reasonable assumption for the consituents of a rigid body):

$$\frac{\mathrm{d}\vec{L}_{\text{sys}}}{\mathrm{d}t} = \vec{\tau}_{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_{i,\text{ext}}$$

Remarkably, even though the above derivation assumed all vectors were measured w.r.t. an inertial frame, the result still holds if the CoM is accelerating, as long as $\vec{L}_{\text{sys}}$ and $\vec{\tau}_{\text{ext}}$ are measured using CoM as origin.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{L}_{\text{sys}}(\text{about CoM}) = \vec{\tau}_{\text{ext}}(\text{about CoM})$$

Also note that Taylor uses the symbol Γ for torque rather than the symbol τ that is more familiar from freshman physics. He also uses $\mathbf{\Gamma}$ instead of $\vec{\Gamma}$ for vectors, because $\dot{\mathbf{\Gamma}}$ looks nicer than $\dot{\vec{\Gamma}}$

Let's see why (in Ch. 10) [reference — too tedious to do in class]

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{L}_{\mathrm{sys}}(\text{about CoM}) \;=\; \vec{\tau}_{\mathrm{ext}}(\text{about CoM})$$

even if CoM is accelerating. Let $\vec{r'}_i = \vec{r}_i - \vec{R}_{\mathrm{cm}}$. Then

$$\vec{r}_i = \vec{r'}_i + \vec{R}_{\mathrm{cm}} \qquad \vec{v}_i = \vec{v'}_i + \vec{V}_{\mathrm{cm}} \qquad\qquad \vec{R'}_{\mathrm{cm}} \;=\; \vec{0} \;=\; \vec{V'}_{\mathrm{cm}}$$

Assume unprimed frame is inertial, but $\vec{R}_{\mathrm{cm}}$ may be accelerating.

$$\vec{L}_{\mathrm{sys}} \;=\; \sum_i \vec{r}_i\times\vec{p}_i \;=\; \sum_i \vec{r}_i\times m_i\vec{v}_i \;=\; \sum_i(\vec{r'}_i+\vec{R}_{\mathrm{cm}})\times m_i(\vec{v'}_i+\vec{V}_{\mathrm{cm}}) \;=$$

$$\sum_i \vec{r'}_i\times m_i\vec{v'}_i + \Big(\sum_i m_i\vec{r'}_i\Big)\times\vec{V}_{\mathrm{cm}} + \vec{R}_{\mathrm{cm}}\times\Big(\sum_i m_i\vec{v'}_i\Big) + \vec{R}_{\mathrm{cm}}\times\Big(\sum_i m_i\Big)\vec{V}_{\mathrm{cm}}$$

$$= \sum_i \vec{r'}_i\times m_i\vec{v'}_i + \vec{R'}_{\mathrm{cm}}\times\vec{V}_{\mathrm{cm}} + \vec{R}_{\mathrm{cm}}\times(M_{\mathrm{tot}}\vec{V'}_{\mathrm{cm}}) + \vec{R}_{\mathrm{cm}}\times M_{\mathrm{tot}}\vec{V}_{\mathrm{cm}}$$

$$= \sum_i \vec{r'}_i\times m_i\vec{v'}_i \;+\; \vec{0}\times\vec{V}_{\mathrm{cm}} \;+\; \vec{R}_{\mathrm{cm}}\times\vec{0} \;+\; \vec{R}_{\mathrm{cm}}\times M_{\mathrm{tot}}\vec{V}_{\mathrm{cm}}$$

$$\vec{L}_{\rm sys} = \vec{R}_{\rm cm} \times M_{\rm tot}\vec{V}_{\rm cm} + \sum_i \vec{r'}_i \times m_i \vec{v'}_i = \vec{L}_{\rm orbital} + \vec{L}_{\rm spin}$$

where $\vec{L}_{\rm spin}$ is "angular momentum of the motion relative to CM"

$$\frac{\rm d}{{\rm d}t}\vec{L}_{\rm sys} = \vec{V}_{\rm cm} \times M_{\rm tot}\vec{V}_{\rm cm} + \vec{R}_{\rm cm} \times M_{\rm tot}\vec{A}_{\rm cm} + \frac{\rm d}{{\rm d}t}(\sum_i \vec{r'}_i \times m_i \vec{v'}_i)$$

$$= \vec{0} + \vec{R}_{\rm cm} \times \vec{F}^{\rm ext,net} + \frac{\rm d}{{\rm d}t}(\sum_i \vec{r'}_i \times m_i \vec{v'}_i)$$

$$\frac{\rm d}{{\rm d}t}\vec{L}_{\rm sys} = \vec{\tau} = \text{“torque \textbf{on} CM”} + \text{“torque \textbf{about} CM”}$$

Now write out $\vec{\tau}$ in terms of $\vec{r'}$ and $\vec{R}_{\rm cm}$

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i^{\rm net,ext} = \sum_i (\vec{R}_{\rm cm} + \vec{r'}_i) \times \vec{F}_i^{\rm net,ext}$$

$$\vec{\tau} = \vec{R}_{\rm cm} \times (\sum_i \vec{F}_i^{\rm net,ext}) + \sum_i \vec{r'}_i \times \vec{F}_i^{\rm net,ext}$$

$$\frac{\rm d}{{\rm d}t}(\sum_i \vec{r'}_i \times m_i \vec{v'}_i) = \sum_i \vec{r'}_i \times \vec{F}_i^{\rm net,ext} \Rightarrow \boxed{\frac{\rm d}{{\rm d}t}\vec{L}_{\rm spin} = \vec{\tau}(\text{about CM})}$$

If you go through those last several slides at home (only if you're curious), you'll see that we **did not** need to invoke Newton's 2nd law inside a non-inertial frame in order to prove that (even if the CoM is accelerating),

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{L}_{\rm sys}(\text{about CoM}) \;=\; \vec{\tau}_{\rm ext}(\text{about CoM})$$

as long as $\vec{L}_{\rm sys}$ and $\vec{\tau}_{\rm ext}$ are measured using CoM as origin.

A more intuitive way to see this is to imagine putting a turn-table on an accelerating train.

If the rotation axis coincides with the CoM of the turn-table, then the pseudo-force ("inertial force") due to the train's acceleration has zero lever-arm to rotate the turn-table. [We'll see in Ch.9.]

But if rotation axis does not pass through CoM, then the train's acceleration **does** have a non-zero lever arm about this axis.

Chapter 3 asks you to recall, without proof, some basic results from freshman physics about rigid-body rotation about a fixed axis, which we'll take to be the z axis.

$$\vec{L} = L\hat{z} \qquad L_z = I\omega = I\dot{\phi}$$

$$\tau_z = \frac{\mathrm{d}L_z}{\mathrm{d}t} = I\alpha = I\ddot{\phi}$$

$$I = \sum_i m_i(x_i^2 + y_i^2) = \int (x^2 + y^2)\ \mathrm{d}m$$

and in Chapter 4, kinetic energy $\quad T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

In Chapter 10, we'll generalize to

$$\vec{L} = \underline{\underline{I}} \cdot \vec{\omega} \qquad T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\,\vec{\omega} \cdot \underline{\underline{I}} \cdot \vec{\omega}$$

where $\underline{\underline{I}}$ becomes a matrix, and in general $\vec{L}$ and $\vec{\omega}$ can point along different axes! This has some surprising consequences!

I put this here in case we have extra time:

Consider the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$.

(a) By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

(b) Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$z^2 = e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$$

$$= (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

$$\sin(2\theta) = 2\cos\theta\sin\theta$$

$$z^3 = e^{3i\theta} = \cos(3\theta) + i\sin(3\theta)$$

$$= (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$$

```
TrigExpand[Cos[3 θ]]

]= Cos[θ]^3 - 3 Cos[θ] Sin[θ]^2

]:= TrigExpand[Sin[3 θ]]

]= 3 Cos[θ]^2 Sin[θ] - Sin[θ]^3
```

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