Physics 351 — Friday, January 19, 2018

Work on this while you wait for your classmates to arrive:

Consider the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$.

(a) By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

(b) Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$z^2 = e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$$

$$= (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

$$\sin(2\theta) = 2\cos\theta\sin\theta$$

$$z^3 = e^{3i\theta} = \cos(3\theta) + i\sin(3\theta)$$

$$= (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$$

```
TrigExpand[Cos[3 θ]]
```

```
Cos[θ]^3 - 3 Cos[θ] Sin[θ]^2
```

```
TrigExpand[Sin[3 θ]]
```

```
3 Cos[θ]^2 Sin[θ] - Sin[θ]^3
```

Physics 351 — Friday, January 19, 2018

- You've now read Chapters 1–5.
- We're flying through review chapters 1–5 so that we can spend more of the semester on the new material, as last year's students suggested. The pace will calm down next week.
- For Monday, read Chapter 6 (Calculus of Variations), which is the first "new" topic, though some parts of Chapter 5 are probably also new to you. Calculus of variations is a generalization of calculus that you may find somewhat mind-blowing. It is the mathematical underpinning of the Lagrangian formulation of mechanics.
- Homework #1 due on Friday 1/26. Handing out now.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- We'll spend today on Chapter 4 (Energy), with a segue into Chapter 5 (Oscillations). We'll spend next week on Ch 5–6. Lagrangians by the end of next Friday.

Chapter 4

The work done on a particle by **net force** $\vec{F}$ as it moves from point 1 to point 2 equals the change in the particle's KE:

$$\Delta T \ = \ \Delta(\frac{1}{2}mv^2) \ = \ \int_1^2 \vec{F}^{\rm net} \cdot {\rm d}\vec{r} \ \equiv \ W(1 \to 2)$$

If every force $\vec{F}_i$ acting on the particle is **conservative**, then a **potential energy** $U_i(\vec{r})$ can be defined for each $\vec{F}_i$, and the total mechanical energy is constant:

$$E \ = \ T + U_1 + U_2 + \cdots + U_n \ = \ \text{constant}$$

Or more generally, $\Delta E = W_{\rm nc}$. Change in mechanical energy equals the work done by non-conservative forces.

This is important because the Lagrangian formalism only works well with forces that can be derived from a potential energy $U(\vec{r})$.

A force $\vec{F}$ is **conservative** if

(a) it depends only on the particle's position: $\vec{F} = \vec{F}(\vec{r})$

(b) for any two points 1 and 2, the work $W(1 \to 2)$ done by $\vec{F}$ is the same for all paths joining 1 and 2.

Condition (b) is equivalent to the vector-calculus statement $\nabla \times \vec{F} = 0$, i.e. curl of $\vec{F}$ is zero.

In one dimension, condition (a) implies condition (b).

If $\vec{F}$ is conservative, you can define **potential energy**

$$U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r'}) \cdot \mathrm{d}\vec{r'}$$

so then the force is minus the gradient of the potential energy:

$$\vec{F} = -\nabla U$$

i.e. $F_x = -\partial U/\partial x$, $F_y = -\partial U/\partial y$, $F_z = -\partial U/\partial z$.

Notice that $\nabla \times (-\nabla U) = \vec{0}$

Notice that $\nabla \times (-\nabla U) = \vec{0}$

$$\vec{\nabla} \times (-\vec{\nabla} U) = \sum_{ijk} \nabla_i (-\vec{\nabla} U)_j \, \hat{e}_k \, \varepsilon_{ijk}$$

$$= \sum_{ijkl} \nabla_i \left(-\nabla_l U \, \hat{e}_l\right)_j \hat{e}_k \, \varepsilon_{ijk}$$

$$= \sum_{ijk} \nabla_i \left(-\nabla_j U\right) \hat{e}_k \, \varepsilon_{ijk}$$

$$= -\sum_{ijk} \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} U\right) \hat{e}_k \, \varepsilon_{ijk}$$

$$= -\left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right)\hat{z} - \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right)\hat{x} - \left(\frac{\partial^2 U}{\partial z \partial x} - \frac{\partial^2 U}{\partial x \partial z}\right)\hat{y}$$

$$= (0, 0, 0)$$

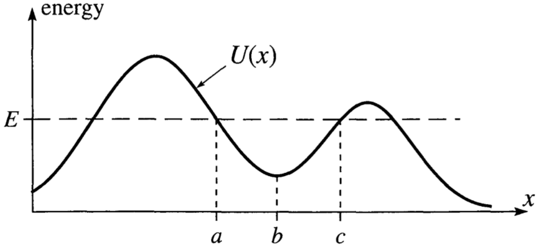
If you're intrigued by this funny ϵ_{ijk} notation, I put some slides on the topic at the back of today's slides. It's called "Cartesian Einstein notation."

Reminding yourself what makes a force conservative is worthwhile here because the Lagrangian approach doesn't work (at least not cleanly) with non-conservative forces such as sliding friction, drag forces, etc.

Non-conservative forces of constraint (which don't do any work), such as magnetic forces, or static friction, or the "normal force" of contact, or the tension in a rigid rod, etc., will turn out to be OK.

The weird case that $\vec{F}(\vec{r},t) = -\nabla U(\vec{r},t)$ where $\vec{F}$ can be derived from a *time-dependent* potential energy $U(\vec{r},t)$ is a case that the Lagrangian formalism has no trouble with, which is why Taylor mentions it in Chapter 4.

Life is especially simple for a conservative force in one dimension. In Chapter 8, we'll reduce the Kepler problem to a 1D problem, which **tremendously simplifies** the solution.

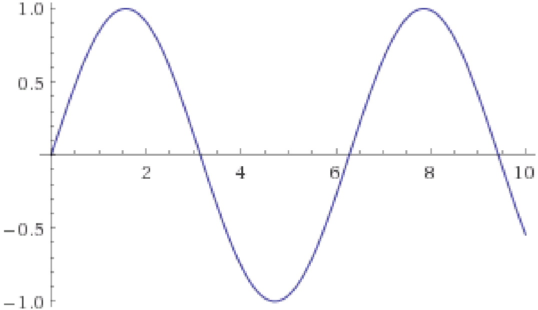


How many equilibrium points do you see here?

How many of them are stable?

How would you find the frequency of small oscillations about the stable one(s)?

For what range of energies is the particle "bound," i.e. unable to escape from the nearby region?



For $U(x) = \sin(x)$, where is the force largest?

So you can read off a lot of information by looking at $U(x)$ for a 1D conservative system. You can even "read off" the EOM:

$$\frac{1}{2}mv^2 \;=\; T \;=\; E^{\mathrm{mech}} - U(x)$$

$$\dot{x}(x) = \pm\sqrt{2/m}\,\sqrt{E - U(x)}$$

So you can find $t(x)$ by integration

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x'}{\dot{x}(x')} = \sqrt{\frac{m}{2}}\,\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{E - U(x')}}$$

and then solve to get $x(t)$.

On one upcoming homework problem, you'll use this technique to find the oscillation period of a pendulum for large amplitude.

On another upcoming problem, you'll simply write $\mathrm{d}E/\mathrm{d}t = 0$ to find the EOM for a 1D conservative system.

So systems constrained to one "degree of freedom" are usually easy to solve, even if the motion takes place in 2 or 3 dimensions.

Remember that the definitions of **equilibrium** and **stability** for mechanics may be different from definitions seen in e.g. a dynamical-systems course.

Equilibrium:

$$\vec{0} = \vec{F} = -\nabla U$$

equilibrium in 1D: $\mathrm{d}U/\mathrm{d}x = 0$

Stable in 1D:

$$\mathrm{d}^2U/\mathrm{d}x^2 > 0$$

(Stable in 3D: eigenvalues of 3×3 "Hessian" matrix of 2nd partial derivatives are all positive, i.e. $U(\vec{r})$ is at a local minimum in every direction, not a saddle point.)

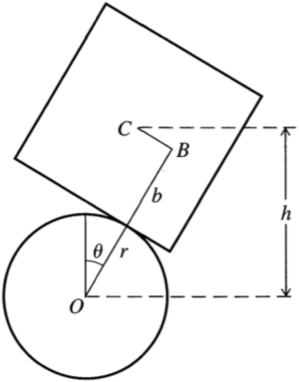


Figure 4.14 A cube, of side $2b$ and center C, is placed on a fixed horizontal cylinder of radius r and center O. It is originally put so that C is centered above O, but it can roll from side to side without slipping.

Textbook states: $U(\theta) = mg((r+b)\cos\theta + r\theta\sin\theta)$

You can work this out in detail on a future XC problem if you wish.

$U'(0) = 0$ for any b/r.

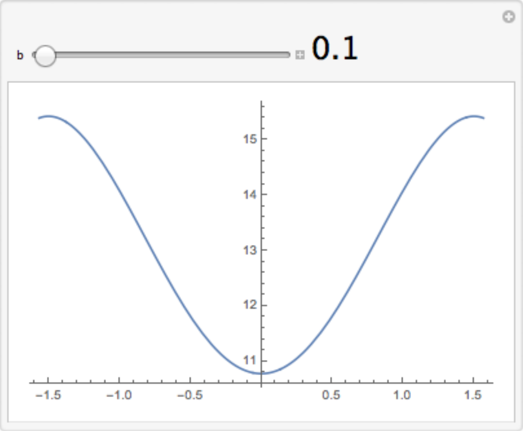
But $U''(0) > 0$ only for $b/r < 1$.

Small cube is stable on top, but big cube is not.

```
ClearAll["Global`*"];

m = 1.0;
g = 9.8;
r = 1.0;
U[theta_, b_] := m g ((r + b) Cos[theta] + r theta Sin[theta]);

Manipulate[
 Plot[U[theta, b], {theta, -Pi / 2, Pi / 2}],
 {b, 0.1, 1.5, Appearance → "Labeled", LabelStyle → Large}
]
```

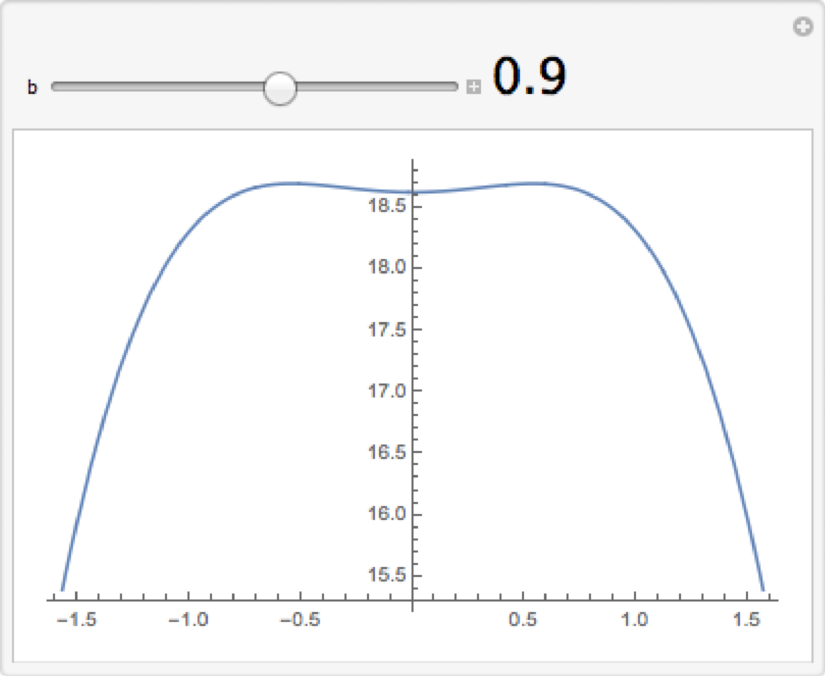


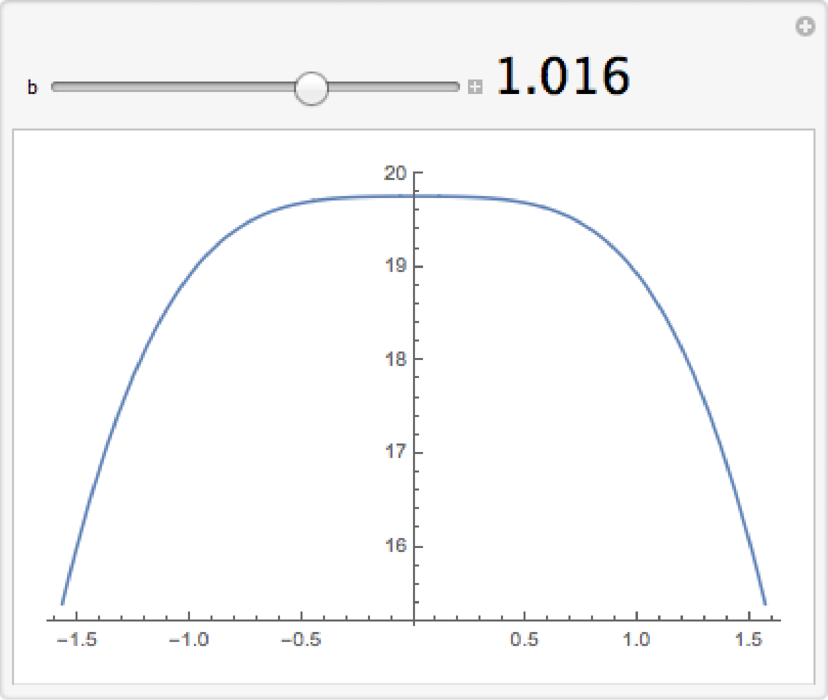
Vary ratio b/r

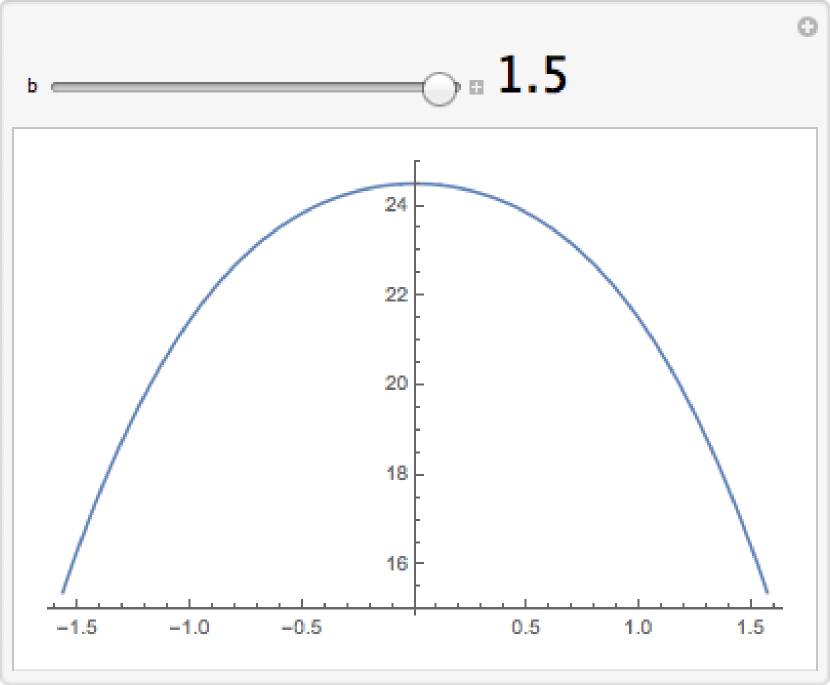
(cube half-edge) / (cylinder radius)

in range
$0.1 \le b/r \le 1.5$

$$U(\theta) = mg((r+b)\cos\theta + r\theta\sin\theta)$$


b
0.9
18.5
18.0
17.5
17.0
16.5
16.0
15.5
−1.5
−1.0
−0.5
0.5
1.0
1.5


b
1.016
20
19
18
17
16
−1.5
−1.0
−0.5
0.5
1.0
1.5


b
1.5
24
22
20
18
16
−1.5
−1.0
−0.5
0.5
1.0
1.5

```
:= ClearAll["Global`*"];

m = 1.0;
g = 9.8;
r = 1.0;
U[theta_, b_] := m g ((r + b) Cos[theta] + r theta Sin[theta]);

Manipulate[
 Plot[U[theta, b], {theta, -Pi / 2, Pi / 2}],
 {b, 0.1, 1.5, Appearance → "Labeled", LabelStyle → Large}
]
```

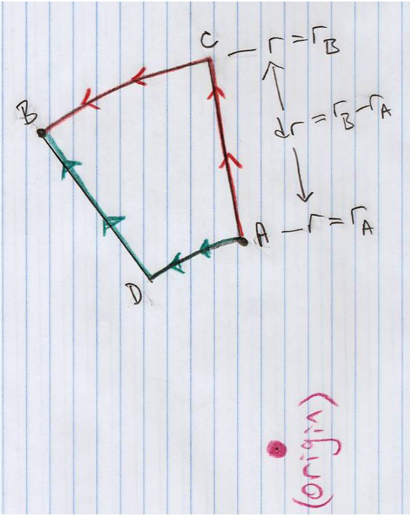
The familiar Coulomb and gravitational forces are both **conservative** and **spherically symmetric**. Remarkably, for central forces, these two properties always go together.

In another future XC problem, you can (if you wish) prove that a central force that is spherically symmetric, i.e.

$$\vec{F}(\vec{r}) \;=\; f(|\vec{r}|)\,\hat{r} \;=\; f(r)\,\hat{r}$$

is automatically conservative. Let's prove the converse: that a central force that is conservative must be spherically symmetric.

Here's an intuitive proof:



Consider paths ACB and ADB, where AC and DB are radial, and AD and CB are arcs of constant radius (r_A or r_B).

Since $\vec{F}$ is conservative, work $W_{A\to C\to B} = W_{A\to D\to B}$.

Since $\vec{F}$ is central $\boxed{\vec{F} = f(\vec{r})\,\hat{r}}$ and thus has no non-radial component, $W_{A\to D} = 0 = W_{C\to B}$.

So $W_{A\to C} = W_{D\to B}$, or $f(\vec{r}_A)\mathrm{d}r = f(\vec{r}_D)\mathrm{d}r$.

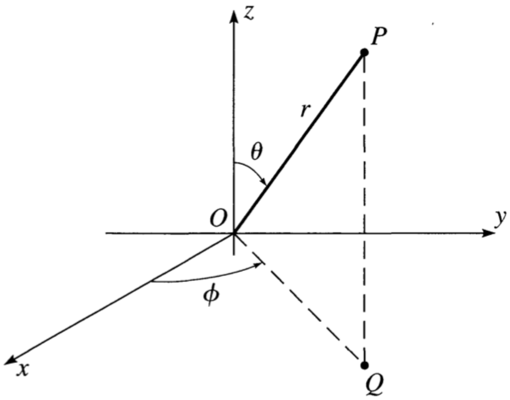
Since A and D can be any two points at the same distance from the origin, $f(\vec{r})$ must depend only on $|\vec{r}|$, i.e. $f(\vec{r}) = f(r)$.

Therefore $\vec{F}(\vec{r}) = \vec{F}(r)$, i.e. $\vec{F}$ is spherically symmetric.

So by chopping up an arbitrary path into segments that are either purely radial or purely tangential, and using the fact that no work is done along the non-radial segments, one proves that if a central force is conservative, then it must also be spherically symmetric, i.e. $F(r, \theta, \phi) = F(r)$.

The textbook shows a different proof, which uses spherical polar coordinates, whose conventions people often find challenging to remember.

So let's jog your memory of spherical polar coordinates.



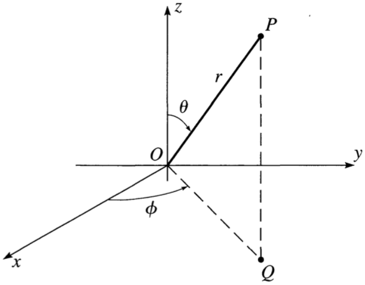
z equals r times what? **(Don't say it out loud yet!)**

x equals r times what times what?

y equals r times what times what?

Standing on Earth, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ point which ways?

$z = r\cos\theta \qquad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$

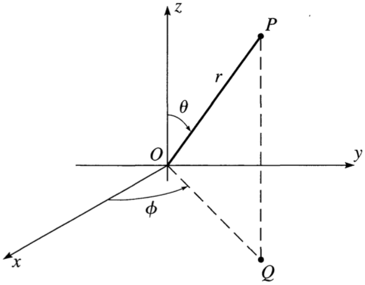


Taylor makes the surprising comment that in polar coordinates,

$$\vec{a}\cdot\vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

What does that mean? (Think ...)

$z = r\cos\theta \qquad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$



Taylor makes the surprising comment that in polar coordinates,

$$\vec{a}\cdot\vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

What does that mean? (Think ...) Could it possibly mean that

$$\text{(No)} \qquad \vec{r}_1\cdot\vec{r}_2 = r_1 r_2 + \theta_1\theta_2 + \phi_1\phi_2 \qquad \text{(No)}$$

(No!)

Taylor's comment that in polar coordinates,

$$\vec{a} \cdot \vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

means that e.g. at one point on or near Earth's surface, you can set up an orthonormal **local** coordinate system and write

$\hat{r}$ = "up" unit vector

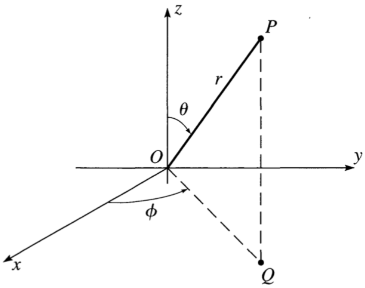
$\hat{\theta}$ = "south" unit vector

$\hat{\phi}$ = "east" unit vector

Then I can write out the components of e.g. a force $\vec{F}$ and a displacement $\Delta\vec{r}$ in that orthonormal coordinate system and write e.g. Work $= \vec{F} \cdot \Delta\vec{r}$

$$W \;=\; F_{\rm up}\Delta r_{\rm up} \;+\; F_{\rm south}\Delta r_{\rm south} \;+\; F_{\rm east}\Delta r_{\rm east}$$

$z = r\cos\theta \quad x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi$

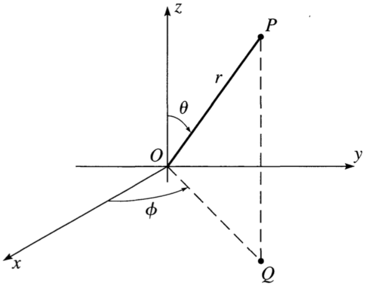


If I move $\mathrm{d}r$ "up," and $\mathrm{d}\theta$ "south," and $\mathrm{d}\phi$ "east," what is my resulting displacement vector?

$$\mathrm{d}\vec{r} \;=\; \hat{r}A + \hat{\theta}B + \hat{\phi}C$$

What are A, B, and C? (All have dimensions of length.)

$z = r\cos\theta \quad x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi$



If I move $\mathrm{d}r$ "up," and $\mathrm{d}\theta$ "south," and $\mathrm{d}\phi$ "east," my displacement vector is

$$\mathrm{d}\vec{r} \;=\; \hat{r}\,\mathrm{d}r \,+\, \hat{\theta}\,r\mathrm{d}\theta \,+\, \hat{\phi}\,r\sin\theta\mathrm{d}\phi$$

(This is useful when computing the distance between two (nearby) terrestrial points, given their (latitude,longitude) geocodes.)

For a function $U(x, y, z)$, we can write (for infinitesimal displacement (dx, dy, dz))

$$U(x+dx, y+dy, z+dz) =$$
$$U(x,y,z) + \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

i.e. the infinitesimal change in U is

$$dU = U(x+dx, y+dy, z+dz) - U(x,y,z)$$
$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

In Cartesian coordinates, we write the gradient of U as

$$\vec{\nabla} U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z}$$

In Cartesian coordinates, we write the gradient of U as

$$\vec{\nabla} U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z}$$

The gradient "points in the direction of steepest ascent." When I take a small step

$$d\vec{r} = \hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz$$

the corresponding change in U is

$$dU = (\vec{\nabla} U) \cdot d\vec{r}$$

$$= (\vec{\nabla} U)_x (d\vec{r})_x + (\vec{\nabla} U)_y (d\vec{r})_y + (\vec{\nabla} U)_z (d\vec{r})_z$$

$$= \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

Treating $\mathcal{U}$ as a function of the 3 variables r, θ, ϕ, we can write

$$d\mathcal{U} = \mathcal{U}(r+dr, \theta+d\theta, \phi+d\phi) - \mathcal{U}(r,\theta,\phi),$$

$$d\mathcal{U} = \frac{\partial \mathcal{U}}{\partial r} dr + \frac{\partial \mathcal{U}}{\partial \theta} d\theta + \frac{\partial \mathcal{U}}{\partial \phi} d\phi$$

We would like also to write

$$d\mathcal{U} = \vec{\nabla}\mathcal{U} \cdot d\vec{r}$$

where $d\vec{r}$ is the displacement corresponding to an infinitesimal step in $\hat{r}, \hat{\theta}, \hat{\phi}$ directions:

$r \to r + dr$: displacement $dr\,\hat{r}$

$\theta \to \theta + d\theta$: displacement $r\,d\theta\,\hat{\theta}$

$\phi \to \phi + d\phi$: displacement $r\sin\theta\,d\phi\,\hat{\phi}$

$$\Rightarrow \quad d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

$$\Rightarrow \quad d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

$$dU = (\vec{\nabla}U)_r (d\vec{r})_r + (\vec{\nabla}U)_\theta (d\vec{r})_\theta + (\vec{\nabla}U)_\phi (d\vec{r})_\phi$$

$$dU = \frac{\partial U}{\partial r}\,dr + \frac{\partial U}{\partial \theta}\,d\theta + \frac{\partial U}{\partial \phi}\,d\phi$$

equating corresponding terms $\Rightarrow$

(You try it!)

$$\Rightarrow \quad d\vec{r} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\phi}\, r \sin\theta\, d\phi$$

$$dU = (\vec{\nabla} U)_r (d\vec{r})_r + (\vec{\nabla} U)_\theta (d\vec{r})_\theta + (\vec{\nabla} U)_\phi (d\vec{r})_\phi$$

$$dU = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \theta} d\theta + \frac{\partial U}{\partial \phi} d\phi$$

equating corresponding terms $\Rightarrow$

$$(\vec{\nabla} U)_r = \frac{\partial U}{\partial r} \qquad (\vec{\nabla} U)_\theta = \left(\frac{1}{r}\right) \frac{\partial U}{\partial \theta} \qquad (\vec{\nabla} U)_\phi = \left(\frac{1}{r \sin\theta}\right) \frac{\partial U}{\partial \phi}$$

$$\Rightarrow \vec{\nabla} U = \hat{r} \frac{\partial U}{\partial r} + \hat{\theta} \left(\frac{1}{r}\right) \frac{\partial U}{\partial \theta} + \hat{\phi} \left(\frac{1}{r \sin\theta}\right) \frac{\partial U}{\partial \phi}$$

Now back to our central force $\vec{F}$ that is known to be conservative. Can we prove that it must also be spherically symmetric?

conservative $\Rightarrow$ $\vec{F} = -\vec{\nabla} U$

$$\Rightarrow \vec{F} = -\hat{r}\frac{\partial U}{\partial r} - \hat{\theta}\left(\frac{1}{r}\right)\frac{\partial U}{\partial \theta} - \hat{\phi}\left(\frac{1}{r\sin\theta}\right)\frac{\partial U}{\partial \phi}$$

central $\Rightarrow$ $\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r} = -\hat{r}\frac{\partial U}{\partial r}$

So the $\hat{\theta}$ and $\hat{\phi}$ terms must be zero everywhere.

$$\Rightarrow \frac{\partial U}{\partial \theta} = 0 = \frac{\partial U}{\partial \phi} \quad \boxed{\text{everywhere}}$$

(A central force exerted by Earth's center can have an up/down component but cannot have E/W or N/S components.)

$$\Rightarrow \quad \frac{\partial U}{\partial \theta} = 0 = \frac{\partial U}{\partial \phi} \qquad \boxed{\text{everywhere}}$$

$$\Rightarrow \quad U = U(r)$$

U is a function only of radius.

$$f(\vec{r}) = -\frac{\partial U}{\partial r} = -\frac{\partial}{\partial r} U(r)$$

is also a function only of radius (no θ or ϕ dependence)

$$\Rightarrow \quad f(\vec{r}) = f(r)$$

$$\vec{F}(\vec{r}) = f(r)\,\hat{r}$$

which is spherically symmetric.

(You can prove the converse as a future XC problem, if you wish.)

Speaking of spherical polar coordinates

Show that the moment of inertia of a uniform solid sphere rotating about a diameter is $I = \frac{2}{5}MR^2$.

The integral is easiest in spherical polar coordinates, with the axis of rotation taken to be the z axis.

Helpful hint: $\mathrm{d}V = r^2\,\mathrm{d}r\ \mathrm{d}(\cos\theta)\ \mathrm{d}\phi$.

For this problem, that form is simpler to use than the other form you may have seen, $\mathrm{d}V = r^2\,\mathrm{d}r\ \sin\theta\,\mathrm{d}\theta\ \mathrm{d}\phi$ But to account for the minus sign you then integrate from $\cos\theta = -1$ to $\cos\theta = +1$ instead of from $\theta = 0$ to $\theta = \pi$.

Go ahead and try it, with your neighbor. I enjoyed working it out on my train ride. (We may skip this if time is short.)

$$I = \int (x^2+y^2)\, dm$$

$$\left.\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \end{array}\right\} \Rightarrow \begin{array}{l} x^2+y^2 = \\ r^2\sin^2\theta \end{array}$$

$$dm = \frac{M}{V}\, dV = \frac{M}{\frac{4}{3}\pi R^3}\, dV = \frac{3M}{4\pi R^3}\, dV$$

$$I = \int r^2\sin^2\theta \left(\frac{3M}{4\pi R^3}\right) dV$$

$$= \frac{3M}{4\pi R^3}\int (r^2\sin^2\theta)\, r^2 dr\; d\cos\theta\; d\phi$$

$$= \frac{3M}{4\pi R^3}\int_0^R r^4 dr \int_{-1}^{+1} (1-\cos^2\theta)\, d\cos\theta \int_0^{2\pi} d\phi$$

$$= \frac{3M}{4\pi R^3}\left[\frac{r^5}{5}\right]_0^R \left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{\cos\theta=-1}^{\cos\theta=+1} \left[\phi\right]_0^{2\pi}$$

$$= \frac{3M}{4\pi R^3}\left(\frac{R^5}{5}\right)\left(1-\frac{1}{3}-(-1)+\left(-\frac{1}{3}\right)\right)(2\pi)$$

$$= \frac{3M}{4\pi R^3}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \boxed{\frac{2}{5}MR^2}$$

This is in the notes for reference. It is too tedious to go through in class, but it's a useful trick to know how to use, if you're interested.

By the way, there is a fun (and at first glance slightly mysterious) way to prove the dreaded "BAC-CAB rule," using the "Cartesian Einstein notation."

Cartesian Einstein notation

vector $\vec{r} = (x, y, z) = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \sum_i r_i \hat{e}_i$

the i^{th} component $(i \in \{1,2,3\})$ of vector $\vec{A}$ is A_i

Kronecker delta : $\delta_{ij} = 1$ if $i = j$, else 0

dot product: $\vec{A} \cdot \vec{B} = \sum_{ij} A_i B_j \delta_{ij} = \sum_i A_i B_i$

matrix • vector : $\underline{\underline{M}} \cdot \vec{r} = \sum_{ij} M_{ij} r_j \hat{e}_i$

$$(\underline{\underline{M}} \cdot \vec{r})_i = \sum_j M_{ij} r_j$$

matrix multiply : $(\underline{\underline{M}} \cdot \underline{\underline{N}})_{ij} = \sum_k M_{ik} N_{kj}$

positron.hep.upenn.edu/p351/files/0119_cartesian_einstein.pdf

Levi-Civita symbol (a.k.a. permutation symbol, antisymmetric symbol)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \in \{123, 231, 312\} \\ -1 & \text{if } ijk \in \{213, 321, 132\} \\ 0 & \text{otherwise} \end{cases}$$

So $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

all others are zero.

Cross product: $\vec{A} \times \vec{B} = \sum_{ijk} A_i B_j \hat{e}_k \, \varepsilon_{ijk}$

$$\vec{A} \times \vec{B} = (A_1 B_2 - A_2 B_1)\hat{e}_3 + (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$$

$$(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$$

Incredibly useful identity:

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Now use it to prove the dreaded "BAC−CAB" rule:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \sum_{ijk} A_i (\vec{B}\times\vec{C})_j \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklmn} A_i (B_l C_m \hat{e}_n \varepsilon_{lmn})_j \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i (B_l C_m \varepsilon_{lmj}) \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k (\varepsilon_{lmj}\varepsilon_{ijk})$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k \left(\varepsilon_{lmj} \varepsilon_{ijk} \right)$$

$$\varepsilon_{lmj} \varepsilon_{ijk} = \varepsilon_{lmj} \varepsilon_{kij} = \left(\delta_{lk} \delta_{mi} - \delta_{li} \delta_{mk} \right)$$

$$= \sum_{ijklm} \left(A_i B_l C_m \hat{e}_k \delta_{lk} \delta_{mi} - A_i B_l C_m \hat{e}_k \delta_{li} \delta_{mk} \right)$$

$$= \sum_{ik} \left(A_i B_k C_i \hat{e}_k - A_i B_i C_k \hat{e}_k \right)$$

$$= \left(\sum_i A_i C_i \right) \left(\sum_k B_k \hat{e}_k \right) - \left(\sum_i A_i B_i \right) \left(\sum_k C_k \hat{e}_k \right)$$

$$= \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C}$$

$$= \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

I asked you for last night's reading (optionally) to try this experiment described by Taylor in Ch 5, footnote 14, page 192:

"The behavior of [the phase shift] δ can, nevertheless, be observed. Make a simple pendulum from a piece of string and a metal nut, and drive it by holding it at the top and moving your hand from side to side. The most obvious thing is that you will be most successful at driving it when your frequency equals the natural frequency, but you can also see that when you drive more slowly the pendulum moves in step with your hand, whereas when you move more quickly the pendulum moves oppositely to your hand."

Physics 351 — Friday, January 19, 2018

- You've now read Chapters 1–5.
- We're flying through review chapters 1–5 so that we can spend more of the semester on the new material, as last year's students suggested. The pace will calm down next week.
- For Monday, read Chapter 6 (Calculus of Variations), which is the first "new" topic, though some parts of Chapter 5 are probably also new to you. Calculus of variations is a generalization of calculus that you may find somewhat mind-blowing. It is the mathematical underpinning of the Lagrangian formulation of mechanics.
- Homework #1 due on Friday 1/26. Handing out now.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- We'll spend today on Chapter 4 (Energy), with a segue into Chapter 5 (Oscillations). We'll spend next week on Ch 5–6. Lagrangians by the end of next Friday.