Physics 351 — Monday, January 22, 2018

Phys 351

Work on this while you wait for your classmates to arrive:

Show that the moment of inertia of a uniform solid sphere rotating about a diameter is $I = \frac{2}{5}MR^2$.

The integral is easiest in spherical polar coordinates, with the axis of rotation taken to be the z axis.

Helpful hint: $\mathrm{d}V = r^2\,\mathrm{d}r\;\mathrm{d}(\cos\theta)\;\mathrm{d}\phi$.

[For this problem, that form is simpler to use than the other form you may have seen, $\mathrm{d}V = r^2\,\mathrm{d}r\;\sin\theta\,\mathrm{d}\theta\;\mathrm{d}\phi$ But to account for the minus sign you then integrate from $\cos\theta = -1$ to $\cos\theta = +1$ instead of from $\theta = 0$ to $\theta = \pi$.]

$$I = \int (x^2+y^2)\, dm$$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \end{aligned}\right\} \Rightarrow \begin{aligned} x^2+y^2 &= \\ r^2\sin^2\theta \end{aligned}$$

$$dm = \frac{M}{V}\, dV = \frac{M}{\frac{4}{3}\pi R^3}\, dV = \frac{3M}{4\pi R^3}\, dV$$

$$I = \int r^2\sin^2\theta \left(\frac{3M}{4\pi R^3}\right) dV$$

$$= \frac{3M}{4\pi R^3} \int (r^2\sin^2\theta)\, r^2 dr\, d\cos\theta\, d\phi$$

$$= \frac{3M}{4\pi R^3} \int_0^R r^4 dr \int_{-1}^{+1} (1-\cos^2\theta)\, d\cos\theta \int_0^{2\pi} d\phi$$

$$= \frac{3M}{4\pi R^3} \left[\frac{r^5}{5}\right]_0^R \left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{\cos\theta=-1}^{\cos\theta=+1} \left[\phi\right]_0^{2\pi}$$

$$= \frac{3M}{4\pi R^3} \left(\frac{R^5}{5}\right)\left(1-\frac{1}{3}-(-1)+\left(-\frac{1}{3}\right)\right)(2\pi)$$

$$= \frac{3M}{4\pi R^3}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \boxed{\frac{2}{5}MR^2}$$

Physics 351 — Monday, January 22, 2018

- Homework #1 due on Friday 1/26.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- After finishing up Friday's discussion of spherical polar coordinates, we'll spend the rest of this week on Ch 5–6. I'm aiming to start Lagrangians by the end of Friday.
- You've now read Chapters 1–6. The pace will calm down now, as we start the new material.

Taylor's Chapter 4 comment that in polar coordinates,

$$\vec{a} \cdot \vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

means that e.g. at one point on or near Earth's surface, you can set up an orthonormal **local** coordinate system and write

$\hat{r}$ = "up" unit vector

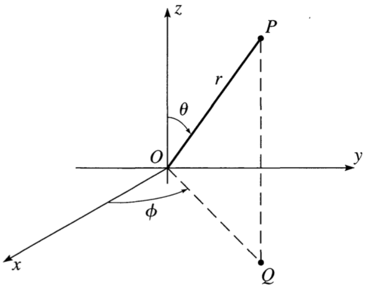
$\hat{\theta}$ = "south" unit vector

$\hat{\phi}$ = "east" unit vector

Then I can write out the components of e.g. a force $\vec{F}$ and a displacement $\Delta\vec{r}$ in that orthonormal coordinate system and write e.g. Work $= \vec{F} \cdot \Delta\vec{r}$

$$W \; = \; F_{\rm up} \Delta r_{\rm up} \; + \; F_{\rm south} \Delta r_{\rm south} \; + \; F_{\rm east} \Delta r_{\rm east}$$

$$z = r\cos\theta \qquad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$

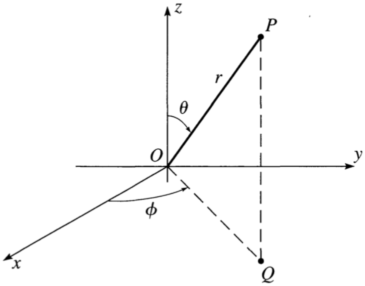


If I move $\mathrm{d}r$ "up," and $\mathrm{d}\theta$ "south," and $\mathrm{d}\phi$ "east," what is my resulting displacement vector?

$$\mathrm{d}\vec{r} \;=\; \hat{r}A + \hat{\theta}B + \hat{\phi}C$$

What are A, B, and C? (All have dimensions of length.)

$$z = r\cos\theta \qquad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$



If I move $\mathrm{d}r$ "up," and $\mathrm{d}\theta$ "south," and $\mathrm{d}\phi$ "east," my displacement vector is

$$\mathrm{d}\vec{r} \;=\; \hat{r}\,\mathrm{d}r \,+\, \hat{\theta}\,r\mathrm{d}\theta \,+\, \hat{\phi}\,r\sin\theta\mathrm{d}\phi$$

(This is useful when computing the distance between two (nearby) terrestrial points, given their (latitude,longitude) geocodes.)

For a function $\mathcal{U}(x, y, z)$, we can write (for infinitesimal displacement (dx, dy, dz))

$$\mathcal{U}(x+dx, y+dy, z+dz) =$$

$$\mathcal{U}(x, y, z) + \frac{\partial \mathcal{U}}{\partial x} dx + \frac{\partial \mathcal{U}}{\partial y} dy + \frac{\partial \mathcal{U}}{\partial z} dz$$

i.e. the infinitesimal change in $\mathcal{U}$ is

$$d\mathcal{U} = \mathcal{U}(x+dx, y+dy, z+dz) - \mathcal{U}(x, y, z)$$

$$d\mathcal{U} = \frac{\partial \mathcal{U}}{\partial x} dx + \frac{\partial \mathcal{U}}{\partial y} dy + \frac{\partial \mathcal{U}}{\partial z} dz$$

In Cartesian coordinates, we write the gradient of $\mathcal{U}$ as

$$\vec{\nabla} \mathcal{U} = \hat{x} \frac{\partial \mathcal{U}}{\partial x} + \hat{y} \frac{\partial \mathcal{U}}{\partial y} + \hat{z} \frac{\partial \mathcal{U}}{\partial z}$$

In Cartesian coordinates, we write the gradient of U as

$$\vec{\nabla} U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}$$

The gradient "points in the direction of steepest ascent." When I take a small step

$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

the corresponding change in U is

$$\begin{aligned} dU &= (\vec{\nabla} U)\cdot d\vec{r} \\ &= (\vec{\nabla} U)_x (d\vec{r})_x + (\vec{\nabla} U)_y (d\vec{r})_y + (\vec{\nabla} U)_z (d\vec{r})_z \\ &= \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz \end{aligned}$$

Treating U as a function of the 3 variables r, θ, ϕ, we can write

$$dU = U(r+dr, \theta+d\theta, \phi+d\phi) - U(r,\theta,\phi),$$

$$dU = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \theta} d\theta + \frac{\partial U}{\partial \phi} d\phi$$

We would like also to write

$$dU = \vec{\nabla} U \cdot d\vec{r}$$

where $d\vec{r}$ is the displacement corresponding to an infinitesimal step in $\hat{r}, \hat{\theta}, \hat{\phi}$ directions:

$r \to r + dr$: displacement $dr\,\hat{r}$

$\theta \to \theta + d\theta$: displacement $r\,d\theta\,\hat{\theta}$

$\phi \to \phi + d\phi$: displacement $r\sin\theta\,d\phi\,\hat{\phi}$

$$\Rightarrow \quad d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

$$\Rightarrow \quad d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

$$dU = (\vec{\nabla}U)_r (d\vec{r})_r + (\vec{\nabla}U)_\theta (d\vec{r})_\theta + (\vec{\nabla}U)_\phi (d\vec{r})_\phi$$

$$dU = \frac{\partial U}{\partial r}\,dr + \frac{\partial U}{\partial \theta}\,d\theta + \frac{\partial U}{\partial \phi}\,d\phi$$

equating corresponding terms $\Rightarrow$

(You try it!)

$$\Rightarrow \quad d\vec{r} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\phi}\, r \sin\theta\, d\phi$$

$$dU = (\vec{\nabla} U)_r (d\vec{r})_r + (\vec{\nabla} U)_\theta (d\vec{r})_\theta + (\vec{\nabla} U)_\phi (d\vec{r})_\phi$$

$$dU = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \theta} d\theta + \frac{\partial U}{\partial \phi} d\phi$$

equating corresponding terms $\Rightarrow$

$$(\vec{\nabla} U)_r = \frac{\partial U}{\partial r} \qquad (\vec{\nabla} U)_\theta = \left(\frac{1}{r}\right) \frac{\partial U}{\partial \theta} \qquad (\vec{\nabla} U)_\phi = \left(\frac{1}{r \sin\theta}\right) \frac{\partial U}{\partial \phi}$$

$$\Rightarrow \vec{\nabla} U = \hat{r} \frac{\partial U}{\partial r} + \hat{\theta} \left(\frac{1}{r}\right) \frac{\partial U}{\partial \theta} + \hat{\phi} \left(\frac{1}{r \sin\theta}\right) \frac{\partial U}{\partial \phi}$$

Now back to our central force $\vec{F}$ that is known to be conservative. Can we prove that it must also be spherically symmetric?

$$\text{conservative} \Rightarrow \quad \vec{F} = -\vec{\nabla} U$$

$$\Rightarrow \vec{F} = -\hat{r}\,\frac{\partial U}{\partial r} - \hat{\theta}\left(\frac{1}{r}\right)\frac{\partial U}{\partial \theta} - \hat{\phi}\left(\frac{1}{r\sin\theta}\right)\frac{\partial U}{\partial \phi}$$

$$\text{central} \Rightarrow \vec{F}(\vec{r}) = f(\vec{r})\,\hat{r} = -\hat{r}\,\frac{\partial U}{\partial r}$$

So the $\hat{\theta}$ and $\hat{\phi}$ terms must be zero everywhere.

$$\Rightarrow \frac{\partial U}{\partial \theta} = 0 = \frac{\partial U}{\partial \phi} \quad \boxed{\text{everywhere}}$$

(A central force exerted by Earth's center can have an up/down component but cannot have E/W or N/S components.)

$$\Rightarrow \quad \frac{\partial U}{\partial \theta} = 0 = \frac{\partial U}{\partial \phi} \qquad \boxed{\text{everywhere}}$$

$$\Rightarrow \quad U = U(r)$$

U is a function only of radius.

$$f(\vec{r}) = -\frac{\partial U}{\partial r} = -\frac{\partial}{\partial r} U(r)$$

is also a function only of radius (no θ or ϕ dependence)

$$\Rightarrow \quad f(\vec{r}) = f(r)$$

$$\vec{F}(\vec{r}) = f(r)\,\hat{r}$$

which is spherically symmetric.

(You can prove the converse as a future XC problem, if you wish.)

One point worth emphasizing from the end of Chapter 4 (Energy):

For two particles interacting only with each other,

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

in this case,

$$\frac{\partial U}{\partial x_1} = -\frac{\partial U}{\partial x_2} \qquad \frac{\partial U}{\partial y_1} = -\frac{\partial U}{\partial y_2} \qquad \frac{\partial U}{\partial z_1} = -\frac{\partial U}{\partial z_2}$$

which implies

$$\vec{F}_2 = -\vec{F}_1$$

Since there is no ext. force, this is just Newton #3: $\vec{F}_{12} = -\vec{F}_{21}$

You know 3rd law $\leftrightarrow$ momentum conservation

Deep connection (Noether's theorem):

translation invariance $\leftrightarrow$ momentum conservation

Damped harmonic motion (b = linear drag coefficient from ch2):

$$m\ddot{x} = -kx - b\dot{x}$$

let $\omega_0 = \sqrt{k/m}$ let $\beta = b/(2m)$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

This is a linear, 2nd order, homogeneous differential equation.

- Linear because x, $\dot{x}$, $\ddot{x}$, etc. appear only as the first power, not e.g. $\dot{x}^2$, $x\dot{x}$, x^2, $\sin(x)$, etc.
- More precisely, "linear" because we're applying a linear operator $\mathcal{D}$ to turn the variable x into the LHS:

$$\mathcal{D} \;=\; \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2$$

$$\mathcal{D}[Ax_1+Bx_2] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1+Bx_2) + 2\beta\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1+Bx_2) + \omega_0^2(Ax_1+Bx_2) =$$

$$A\ddot{x}_1 + 2\beta A\dot{x}_1 + A\omega_0^2 x_1 + B\ddot{x}_2 + 2\beta B\dot{x}_2 + B\omega_0^2 x_2 \;=\; A\mathcal{D}[x_1] + B\mathcal{D}[x_2]$$

The amazingly useful feature of linearity is that linearity permits us to use the $\boxed{\text{superposition principle}}$

- If we have two functions $x_1(t)$ and $x_2(t)$ that separately satisfy

$$\mathcal{D}[x_1(t)] = 0$$

$$\mathcal{D}[x_2(t)] = 0$$

(where $\mathcal{D}$ is a linear operator) then a linear combination $x_3(t) = Ax_1(t) + Bx_2(t)$ will also satisfy

$$\mathcal{D}[x_3(t)] = 0$$

- So the superposition of several solutions is also a solution.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

is a linear, 2nd order, homogeneous differential equation.

- Second-order because the order of the highest derivative is 2.
- Any linear diff eq of order n has n independent solutions, i.e. general solution contains n arbitrary constants.
- Homogeneous because the RHS is zero. If the RHS is some $f(t)$, we call this an "inhomogeneous" diff eq.

- We'll say this again later: The general solution to (inhomogeneous) $\mathcal{D}[x] = f(t)$ is the sum of
  - any particular solution to $\mathcal{D}[x] = f(t)$
  - plus the general solution to $\mathcal{D}[x] = 0$

  We'll need that to study "forced" (or "driven") oscillations.

Now back to our equation.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Let's **guess** (!) a solution $x(t) = Ae^{\alpha t}$ and plug it in:

$$(\alpha^2 + 2\alpha\beta + \omega_0^2)\ Ae^{\alpha t} = 0$$

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

So (except for the degenerate $\beta = \omega_0$ case), we've found our two independent solutions:

$$x(t) = Ae^{-\beta t}e^{+\Omega t} + Be^{-\beta t}e^{-\Omega t}$$

where $\Omega = \sqrt{\beta^2 - \omega_0^2}$. The most common case is "weak damping" ("underdamped"), where $\beta < \omega_0$, so $\Omega^2 < 0$. Then

$$\Omega = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$$

$$x(t) = e^{-\beta t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right) = Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

If $\beta = 0.2\omega_0$, then $\omega_1 \approx 0.98\omega_0$. If $\beta = 0.1\omega_0$, then $\omega_1 \approx 0.995\omega_0$.

By suitable choice of A and B, we can ensure that $x(t)$ is real, and that the arbitrary constants C and ϕ_0 are real.

$$x(t) \;=\; e^{-\beta t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right) \;=\; Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

Digression: $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ $\qquad$ $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$ $\qquad$ $\mathrm{Im}(z) = \frac{1}{2i}(z - z^*)$

By analogy, $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ $\qquad$ $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$

So choosing $A = \frac{1}{2}Ce^{i\phi_0}$ and $B = \frac{1}{2}Ce^{-i\phi_0}$ gives

$$x(t) \;=\; Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

where C and ϕ_0 are fixed by the initial conditions. $\omega_1 (\approx \omega_0)$ and β are properties of the system. "Quality factor" $Q = \omega_0/(2\beta)$.

$$x(t) = Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

$Q = \omega_0/(2\beta)$.

$$\text{energy}(t) \propto e^{-2\beta t} = e^{-\omega_0 t/Q} = e^{-2\pi f_0 t/Q} = e^{-2\pi t/(QT_0)}$$

So after Q periods $(t = QT_0)$, the energy has decreased by a factor $e^{-2\pi} \approx \frac{1}{535} \approx 0.002$.

Equivalently,

$$\frac{Q}{2\pi} = \frac{\text{energy stored in oscillator}}{\text{energy dissipated per cycle}}$$

[First two Mathematica "`Manipulate[]`" demos.]

For the special case $\beta = 0$ ("no damping"), $\omega_1 = \omega_0$:

$$\Omega \ = \ i\sqrt{\omega_0^2 - \beta^2} \ = \ i\sqrt{\omega_0^2 - 0} \ = \ i\omega_0$$

$$x(t) \ = \ Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \ = \ C\cos(\omega_0 t + \phi_0)$$

For the $\beta > \omega_0$ "strong damping" ("overdamped") case, $\Omega^2 > 0$, Ω is real and nonzero: $\Omega = \sqrt{\beta^2 - \omega_0^2}$. Then

$$x(t) = Ae^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + Be^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

The first term dominates the decay rate, since the second term decays away more quickly.

Interestingly, in this "overdamped" regime, increasing β (more damping) actually makes the motion decay **less quickly**!

Decay rate is largest at "critical" damping, $\Omega^2 = 0$. Important for shock absorbers, indicator needles.

Critical damping ($\Omega^2 = 0$): Our previous procedure now gives us only one solution: $\quad -\beta \pm \sqrt{\beta^2 - \omega_0^2} \;=\; -\beta$

$$x(t) = Ae^{-\beta t}$$

There must be a second solution to

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0$$

Let's try another lucky guess:

$$x = Bte^{-\beta t}$$

$$\dot{x} = Be^{-\beta t} - \beta Bte^{-\beta t}$$

$$\ddot{x} = -\beta Be^{-\beta t} - \beta Be^{-\beta t} + \beta^2 Bte^{-\beta t}$$

$$\begin{aligned} \ddot{x} &= -2\beta Be^{-\beta t} + \beta^2 Bte^{-\beta t} \\ 2\beta\dot{x} &= \quad 2\beta Be^{-\beta t} - 2\beta^2 Bte^{-\beta t} \\ \beta^2 x &= \qquad\qquad\qquad\quad \beta^2 Bte^{-\beta t} \end{aligned}$$

which add up to zero. So we have

$$x(t) = (A + Bt)e^{-\beta t}$$

Rate of exponential decay (e.g. $1/\tau$) vs. damping constant β.

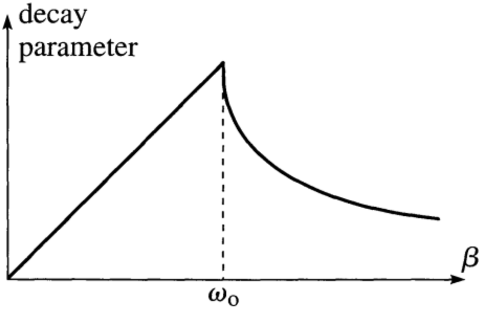


Figure 5.13 The decay parameter for damped oscillations as a function of the damping constant β. The decay parameter is biggest, and the motion dies out most quickly, for critical damping, with $\beta = \omega_o$.

Beyond "critical damping," adding more damping **does not** make the motion decay more quickly!

Driven damped oscillations (why are we allowed to pretend, counterfactually, that the driving force is complex?)

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0 e^{i\omega t}$$

Let's guess a solution

$$x(t) = Ce^{i\omega t}$$

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) Ce^{i\omega t} = F_0 e^{i\omega t}$$

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

$$x(t) = \left(\frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}\right) e^{i\omega t}$$

This is a **particular solution** to the inhomogeneous linear diff. eq.

$$\mathcal{D}[x(t)] = F_0 e^{i\omega t}$$

But we already know that

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})] = 0$$

where $\Omega = \sqrt{\beta^2 - \omega_0^2}$

$$\mathcal{D}[Ce^{i\omega t}] = F_0 e^{i\omega t}$$

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})] = 0$$

So then

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}] = F_0 e^{i\omega t}$$

General solution to (inhomogeneous) $\mathcal{D}[x] = f(t)$ is sum of

- any particular solution to $\mathcal{D}[x] = f(t)$ (inhomogeneous)
- plus the general solution to $\mathcal{D}[x] = 0$ (homogeneous)

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Notice that β and ω_0 (and $\Omega = \sqrt{\beta^2 - \omega_0^2}$) depend only on the oscillator itself, not on the driving force or the initial conditions.

C and ω are properties of the external driving force. A and B depend on initial conditions, but become irrelevant for $t \gg 1/\beta$. (The A and B terms are called the "transient" response.)

For driving force $F_0 e^{i\omega t}$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Once the transients have died away (after $\sim Q$ periods of ω_0),

$$x(t) = Ce^{i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

If the driving force had been $F_0 e^{-i\omega t}$, we would have found

$$x(t) = Ce^{-i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 - 2i\beta\omega + \omega_0^2}$$

Linear superposition lets us average these two solutions to get the response to real driving force $F_0 \cos(\omega t)$.

For driving force $F_0 \cos(\omega t) = \frac{F_0}{2}(e^{i\omega t} + e^{-i\omega t})$, we get (after transients die out)

$$x(t) = \frac{F_0/2}{-\omega^2 + 2i\beta\omega + \omega_0^2}\; e^{i\omega t} + \frac{F_0/2}{-\omega^2 - 2i\beta\omega + \omega_0^2}\; e^{-i\omega t}$$

which is the same as

$$x(t) = \mathrm{Re}\left(\frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}\; e^{i\omega t}\right)$$

which after some algebra is

$$x(t) = A\cos(\omega t - \delta)$$

with

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

[Mathematica and physical demos]

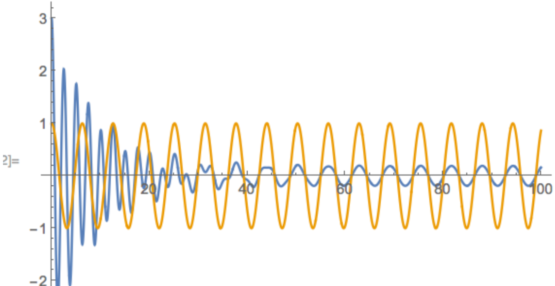
For driving force $F_0 \cos(\omega t)$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + A\cos(\omega t - \delta)$$

with $\Omega = \sqrt{\beta^2 - \omega_0^2} = i\omega_1$

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The important point to remember (for the usual "underdamped" case) is that the transient response rings at $\omega_1 \approx \omega_0$, which is close to the natural frequency, and decays away at rate β. But the long-term response is at the driving frequency ω, with an amplitude and phase that depend on $\omega - \omega_0$.



```
ClearAll["Global`*"];
omega0 = 2.5; omega = 1.0; beta = 0.1; fampl = 1.0; fphi = 0.0; x0 = 3.0;
v0 = 0.0; tmax = 100.0;
soln = NDSolveValue[{x''[t] + 2 beta x'[t] + omega0^2 x[t] == fampl Cos[omega t + fphi],
    x'[0] == v0, x[0] == x0}, x[t], {t, 0, tmax}];
Plot[{soln, fampl Cos[omega t + fphi]}, {t, 0, tmax}, PlotRange → All]
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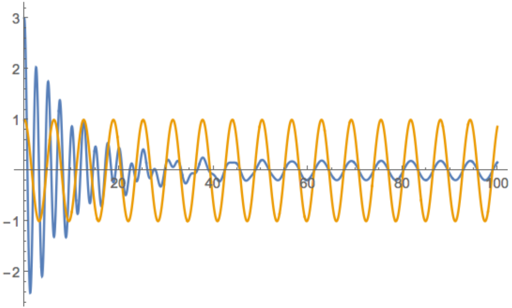


Let's go back to the complex-number driving force

For driving force $F_0 e^{i\omega t}$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Once the transients have died away (after $\sim Q$ periods of ω_0),

$$x(t) = Ce^{i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

Now suppose you have a more complicated driving force:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_a\, e^{i\omega_a t} + F_b\, e^{i\omega_b t}$$

Since $\mathcal{D}$ is linear,

$$\mathcal{D}\left[\frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2}\right] = F_a e^{i\omega_a t}$$

$$\mathcal{D}\left[\frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2}\right] = F_b e^{i\omega_b t}$$

$$\mathcal{D}\left[\frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2} + \frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2}\right] = F_a e^{i\omega_a t} + F_b e^{i\omega_b t}$$

So the general solution is

$$x(t) = \frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2} + \frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2} + e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})$$

where again the transient terms are irrelevant for $t \gg 1/\beta$.

Now consider the more general case

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

and suppose we're able to write

$$f(t) = \sum_n F_n\, e^{i\omega_n t}$$

Then it's clear that the solution would be

$$x(t) = (\text{transient}) + \sum_n \frac{F_n\, e^{i\omega_n t}}{-\omega_n^2 + 2i\beta\omega_n + \omega_0^2}$$

If $f(t)$ is periodic (period $T \equiv 2\pi/\omega$), then Prof. Fourier tells us

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega t}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t}$$

$$\frac{1}{T}\int_{-T/2}^{+T/2} f(t)\, e^{-im\omega t}\, \mathrm{d}t = \sum_n \frac{F_n}{T}\int_{-T/2}^{+T/2} \mathrm{d}t\, e^{i(n-m)\omega t} = \sum_n F_n \delta_{mn} = F_m$$

So the Fourier coefficient F_m is

$$F_m = \frac{1}{T}\int_{-T/2}^{+T/2} f(t) e^{-im\omega t}\, \mathrm{d}t$$

Note: for $f(t)$ real, $F_{-m} = F_m^*$, i.e. the negative-frequency coefficients are the complex conjugates of the corresponding positive-frequency coefficients.

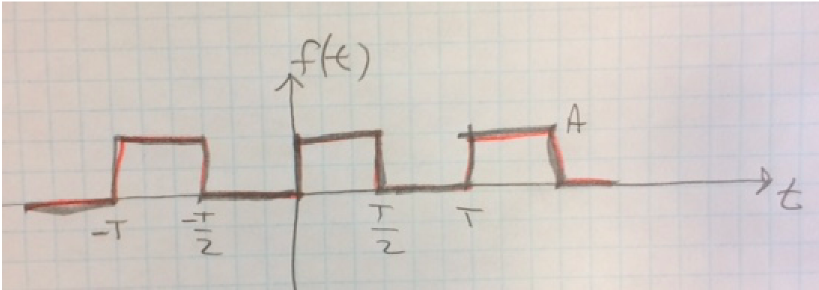
$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t}$$

with Fourier coefficient F_n given by

$$F_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-in\omega t} \, \mathrm{d}t$$

Exercise: use this complex-number Fourier formalism to find the Fourier series for a square wave $f(t)$ of period $T = 2\pi/\omega$, with

$f(t) = 0$ for $-T/2 < t < 0$
$f(t) = A$ for $0 < t < T/2$



Square wave: $f(t) = \begin{cases} 0 & \text{if } -T/2 < t < 0 \\ A & \text{if } 0 < t < T/2 \end{cases}$ $\quad T = \frac{2\pi}{\omega}$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega t}\, dt = \frac{A}{T}\int_0^{T/2} e^{-in\omega t}\, dt = \frac{A}{-in\omega T}\left[e^{-in\omega T/2} - 1\right]$$

$$= \frac{A}{-in2\pi}\left(e^{-in\pi} - 1\right) = \frac{A}{in2\pi}\left(1 - (-1)^n\right)$$

$F_n = 0$ for even $n \neq 0$ $\qquad F_n = \frac{A}{in\pi}$ for odd n

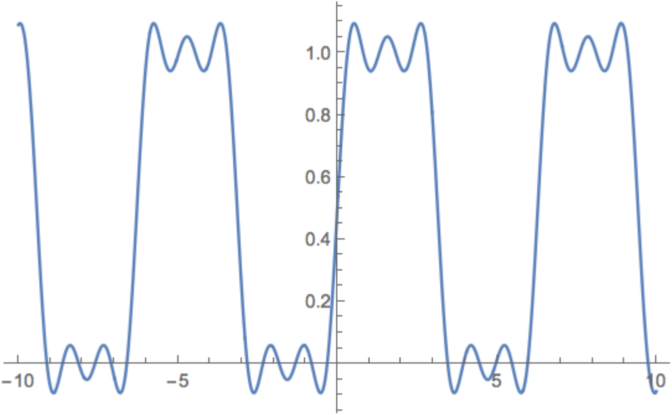
$$F_0 = \frac{A}{T}\int_0^{T/2} dt = \frac{A}{2} = \langle f(t) \rangle$$

$$f(t) = \frac{A}{2} + \sum_{n=1,3,5,\ldots} \frac{A}{in\pi}\underbrace{\left(e^{+in\omega t} - e^{-in\omega t}\right)}_{2i\sin(n\omega t)}$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left(\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \cdots\right)$$

```
699]:= f[t_] := 1 / 2 + (2 / Pi) (Sin[t] + Sin[3 t] / 3 + Sin[5 t] / 5);
       Plot[f[t], {t, -10, 10}]
```

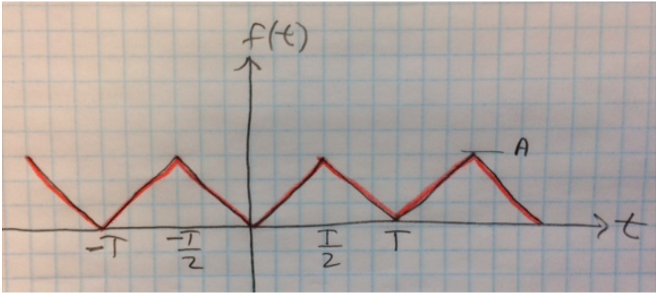
700]=



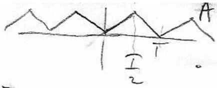
$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t} \qquad\qquad F_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(t) e^{-in\omega t} \, \mathrm{d}t$$

Exercise: use this complex-number Fourier formalism to find the Fourier series for a triangle wave $f(t)$ of period $T = 2\pi/\omega$, with

$f(t) = -2At/T$ for $-T/2 < t < 0$
$f(t) = 2At/T$ for $0 < t < T/2$



$$\text{hint}: \quad \int t e^{-in\omega t} \mathrm{d}t \; = \; \frac{1 + in\omega t}{(n\omega)^2} \; e^{-in\omega t}$$



$$f(t) = \begin{cases} \dfrac{2At}{T} & 0 < t < \dfrac{T}{2} \\ -\dfrac{2At}{T} & -\dfrac{T}{2} < t < 0 \end{cases}$$

$$T = \frac{2\pi}{\omega}$$

$$\omega T = 2\pi$$

$$\omega T/2 = \pi$$

$$F_m = \frac{2A}{T^2}\left(\int_0^{T/2} t e^{-im\omega t}\, dt - \int_{-T/2}^{0} t e^{-im\omega t}\, dt \right)$$

$$= \frac{2A}{T^2}\left(\left[\frac{e^{-im\omega t}(1+im\omega t)}{(m\omega)^2} \right]_0^{T/2} - \left[\frac{e^{-im\omega t}(1+im\omega t)}{(m\omega)^2} \right]_{-T/2}^{0} \right)$$

$$= \frac{2A}{(m\omega T)^2}\left(e^{-im\pi}(1+im\pi) - 1 - 1 + e^{im\pi}(1-im\pi) \right)$$

$$= \frac{2A}{(m2\pi)^2}\left((-1)^m \left(1 + \cancel{im\pi}\right) - 2 + (-1)^m \left(1 - \cancel{im\pi}\right) \right)$$

$$= \frac{A}{(m\pi)^2}\left((-1)^m - 1 \right) \qquad = 0 \text{ for even } m \neq 0$$

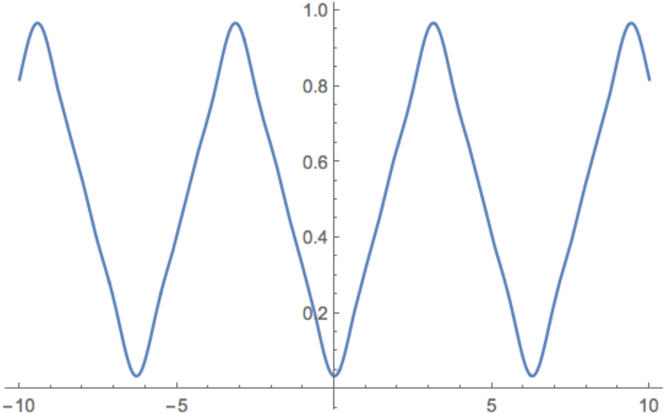
$$\qquad\qquad = \frac{-2A}{(m\pi)^2} \text{ for odd } m$$

$$F_0 = \frac{2A}{T^2}\left(2\int_0^{T/2} t\,dt \right) = \frac{4A}{T^2}\left[\frac{1}{2}\left(\frac{T}{2}\right)^2 \right] = \frac{A}{2} \quad \checkmark$$

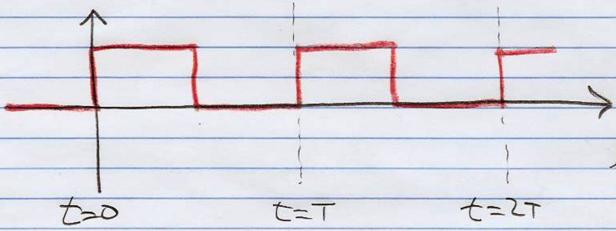
$$f(t) = \frac{A}{2} - \sum_{n=1,3,5,\ldots} \frac{2A}{(n\pi)^2}\Big(\underbrace{e^{in\omega t} + e^{-in\omega t}}_{2\cos(n\omega t)} \Big)$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\left(\cos(\omega t) + \frac{1}{9}\cos(3\omega t) + \frac{1}{25}\cos(5\omega t) \cdots \right)$$

```
f[t_] := 1 / 2 - (4 / Pi ^ 2) (Cos[t] + Cos[3 t] / 9 + Cos[5 t] / 25);
Plot[f[t], {t, -10, 10}]
```



Example: periodic square wave



$$f(t) = \begin{cases} f_A & 0 < t < \frac{T}{2} \\ 0 & -\frac{T}{2} < t < 0 \end{cases}$$

let $\omega = \dfrac{2\pi}{T}$

$$F_m = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)\, e^{-im\omega t} = \frac{f_A}{T}\int_0^{T/2} e^{-im\left(\frac{2\pi}{T}\right)t}\, dt$$

$$= \frac{f_A}{T}\left(\frac{T}{-2\pi i m}\right)\left[e^{-im\left(\frac{2\pi}{T}\right)t}\right]_0^{T/2}$$

$$= -\frac{f_A}{2\pi i m}\left(\left(e^{-i\pi}\right)^m - 1\right) = \frac{f_A}{2\pi i m}\left(1 - (-1)^m\right)$$

For m odd, $F_m = \frac{2f_A}{2\pi i m} = \frac{f_A}{i\pi m}$

For even $m \neq 0$, $F_m = 0$.

For $m = 0$, $F_0 = \frac{f_A}{T}\int_0^{T/2} dt = \frac{f_A}{T}\left(\frac{T}{2}\right) = \boxed{\frac{f_A}{2} = F_0}$

(F_0 is just the average $\langle f(t) \rangle$)

So $x(t) = (\text{transient}) + \sum_n \frac{F_n e^{in\omega t}}{(\omega_0^2 - n^2\omega^2) + 2in\beta\omega}$

Dropping uninteresting transient,

$$x(t) = \sum_n \frac{\left(F_n e^{in\omega t}\right)\left((\omega_0^2 - n^2\omega^2) - 2in\beta\omega\right)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

$$x(t) = \frac{F_0}{\omega_0^2} + \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(e^{in\omega t} - e^{-in\omega t}\right)\left(\omega_0^2 - n^2\omega^2\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

$$+ \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(e^{in\omega t} + e^{-in\omega t}\right)\left(-2in\beta\omega\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(2i\sin(n\omega t)\right)\left(\omega_0^2 - n^2\omega^2\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + \left(2n\beta\omega\right)^2}$$

$$+ \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(2\cos(n\omega t)\right)\left(-2in\beta\omega\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + \left(2n\beta\omega\right)^2}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum \frac{\left(\frac{2f_A}{\pi n}\right) \sin(n\omega t)\,(\omega_0^2 - n^2\omega^2)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

$$+ \sum \frac{\left(\frac{2f_A}{\pi n}\right) \cos(n\omega t)\,(-2n\beta\omega)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

Now let $C_n = \dfrac{\omega_0^2 - n^2\omega^2}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}} \equiv \cos(\delta_n)$

$$S_n = \frac{-2n\beta\omega}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}} \equiv \sin(\delta_n)$$

notice that $C_n^2 + S_n^2 = 1$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{n=1,3,5,\ldots} \sin(n\omega t)\cos(\delta_n)\frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$+ \sum_{n=1,3,5,\ldots} \cos(n\omega t)\sin(-\delta_n)\frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{n=1,3,5,\ldots} A_n \sin(n\omega t - \delta_n)$$

$$\text{where } A_n = \frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$\sin(n\omega t - \delta_n) = \sin(\omega t)\cos(\delta) + \cos(\omega t)\sin(-\delta)$$

$$\delta_n = \arctan\left(\frac{2n\beta\omega}{\omega_0^2 - n^2\omega^2}\right)$$

Notice that the answer came out entirely real, even though we used complex exponentials. Also notice that this looks just like the result from the book using sines and cosines:

$x_n(t) = A_n \cos(n\omega t - \delta_n)$ $\quad A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}}$ $\quad$ same δ_n

and $2f_A/(\pi n)$ is just f_n. ($\sin$ vs. $\cos$ depends on chosen time offset of square wave.)

Physics 351 — Monday, January 22, 2018

- Homework #1 due on Friday 1/26.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- After finishing up Friday's discussion of spherical polar coordinates, we'll spend the rest of this week on Ch 5–6. I'm aiming to start Lagrangians by the end of Friday.
- You've now read Chapters 1–6. The pace will calm down now, as we start the new material.