Physics 351 — Wednesday, January 24, 2018

- Homework #1 due on Friday. Start reading ch7 for Monday.
- I'll hand out HW #2 in class on Friday, and I'll put the PDF online some time tomorrow.
- You can do the Mathematica extra credit any time you like (if at all), but the earlier you do it, the more you'll be able to make use of Mathematica to reduce tedious algebra in your own homework.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- Bill will be in DRL 3N6 on Wednesdays 4pm–7pm
- Grace will be in DRL 2C2 on Thursdays 5:30pm-8:30pm
- We'll spend the rest of this week on Ch 5–6.

Before class: write $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$. Then do $\sin\theta$. Next write $\cosh\theta$ in terms of e^{θ} and $e^{-\theta}$. Then write $\sinh\theta$.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Let's **guess** (!) a solution $x(t) = Ae^{\alpha t}$ and plug it in:

$$(\alpha^2 + 2\alpha\beta + \omega_0^2)\ Ae^{\alpha t} = 0$$

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

So (except for the degenerate $\beta = \omega_0$ case), we've found our two independent solutions:

$$x(t) = Ae^{-\beta t}e^{+\Omega t} + Be^{-\beta t}e^{-\Omega t}$$

where $\Omega = \sqrt{\beta^2 - \omega_0^2}$. The most common case is "weak damping" ("underdamped"), where $\beta < \omega_0$, so $\Omega^2 < 0$. Then

$$\Omega = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$$

$$x(t) = e^{-\beta t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right) = Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

If $\beta = 0.2\omega_0$, then $\omega_1 \approx 0.98\omega_0$. If $\beta = 0.1\omega_0$, then $\omega_1 \approx 0.995\omega_0$.

By suitable choice of A and B, we can ensure that $x(t)$ is real, and that the arbitrary constants C and ϕ_0 are real.

$$x(t) \;=\; e^{-\beta t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right) \;=\; Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

Digression: $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ $\qquad$ $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$ $\qquad$ $\mathrm{Im}(z) = \frac{1}{2i}(z - z^*)$

By analogy, $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ $\qquad$ $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$

So choosing $A = \frac{1}{2}Ce^{i\phi_0}$ and $B = \frac{1}{2}Ce^{-i\phi_0}$ gives

$$x(t) \;=\; Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

where C and ϕ_0 are fixed by the initial conditions. $\omega_1(\approx \omega_0)$ and β are properties of the system. "Quality factor" $Q = \omega_0/(2\beta)$.

$$x(t) = Ce^{-\beta t}\cos(\omega_1 t + \phi_0)$$

$Q = \omega_0/(2\beta)$.

$$\text{energy}(t) \propto e^{-2\beta t} = e^{-\omega_0 t/Q} = e^{-2\pi f_0 t/Q} = e^{-2\pi t/(QT_0)}$$

So after Q periods ($t = QT_0$), the energy has decreased by a factor $e^{-2\pi} \approx \frac{1}{535} \approx 0.002$.

Equivalently,

$$\frac{Q}{2\pi} = \frac{\text{energy stored in oscillator}}{\text{energy dissipated per cycle}}$$

[First two Mathematica "`Manipulate[]`" demos.]
http://positron.hep.upenn.edu/p351/files/0122_ddo.nb
http://positron.hep.upenn.edu/p351/files/0122_ddo.pdf

For the special case $\beta = 0$ ("no damping"), $\omega_1 = \omega_0$:

$$\Omega \;=\; i\sqrt{\omega_0^2 - \beta^2} \;=\; i\sqrt{\omega_0^2 - 0} \;=\; i\omega_0$$

$$x(t) \;=\; Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \;=\; C\cos(\omega_0 t + \phi_0)$$

For the $\beta > \omega_0$ "strong damping" ("overdamped") case, $\Omega^2 > 0$, Ω is real and nonzero: $\Omega = \sqrt{\beta^2 - \omega_0^2}$. Then

$$x(t) = Ae^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + Be^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

The first term dominates the decay rate, since the second term decays away more quickly.

Interestingly, in this "overdamped" regime, increasing β (more damping) actually makes the motion decay **less quickly**!

Decay rate is largest at "critical" damping, $\Omega^2 = 0$. Important for shock absorbers, indicator needles.

Critical damping ($\Omega^2 = 0$): Our previous procedure now gives us only one solution: $\quad -\beta \pm \sqrt{\beta^2 - \omega_0^2} \;=\; -\beta$

$$x(t) = Ae^{-\beta t}$$

There must be a second solution to

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0$$

Let's try another lucky guess:

$$x = Bte^{-\beta t}$$

$$\dot{x} = Be^{-\beta t} - \beta Bte^{-\beta t}$$

$$\ddot{x} = -\beta Be^{-\beta t} - \beta Be^{-\beta t} + \beta^2 Bte^{-\beta t}$$

$$\begin{aligned} \ddot{x} &= -2\beta Be^{-\beta t} + \beta^2 Bte^{-\beta t} \\ 2\beta\dot{x} &= \quad 2\beta Be^{-\beta t} - 2\beta^2 Bte^{-\beta t} \\ \beta^2 x &= \qquad\qquad\qquad\;\; \beta^2 Bte^{-\beta t} \end{aligned}$$

which add up to zero. So we have

$$x(t) = (A + Bt)e^{-\beta t}$$

Rate of exponential decay (e.g. $1/\tau$) vs. damping constant β.

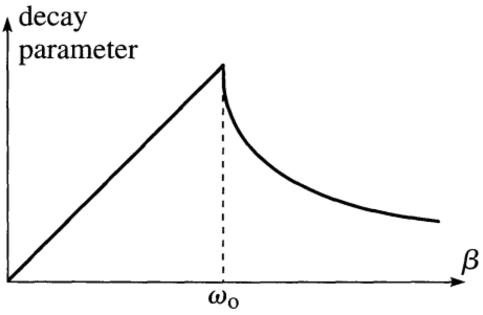


Figure 5.13 The decay parameter for damped oscillations as a function of the damping constant β. The decay parameter is biggest, and the motion dies out most quickly, for critical damping, with $\beta = \omega_o$.

Beyond "critical damping," adding more damping **does not** make the motion decay more quickly!

Driven damped oscillations (why are we allowed to pretend, counterfactually, that the driving force is complex?)

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0 e^{i\omega t}$$

Let's guess a solution

$$x(t) = Ce^{i\omega t}$$

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) Ce^{i\omega t} = F_0 e^{i\omega t}$$

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

$$x(t) = \left(\frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}\right) e^{i\omega t}$$

This is a **particular solution** to the inhomogeneous linear diff. eq.

$$\mathcal{D}[x(t)] = F_0 e^{i\omega t}$$

But we already know that

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})] = 0$$

where $\Omega = \sqrt{\beta^2 - \omega_0^2}$

$$\mathcal{D}[Ce^{i\omega t}] = F_0 e^{i\omega t}$$

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})] = 0$$

So then

$$\mathcal{D}[e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}] = F_0 e^{i\omega t}$$

General solution to (inhomogeneous) $\mathcal{D}[x] = f(t)$ is sum of

- any particular solution to $\mathcal{D}[x] = f(t)$ (inhomogeneous)
- plus the general solution to $\mathcal{D}[x] = 0$ (homogeneous)

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Notice that β and ω_0 (and $\Omega = \sqrt{\beta^2 - \omega_0^2}$) depend only on the oscillator itself, not on the driving force or the initial conditions.

C and ω are properties of the external driving force. A and B depend on initial conditions, but become irrelevant for $t \gg 1/\beta$. (The A and B terms are called the "transient" response.)

For driving force $F_0 e^{i\omega t}$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Once the transients have died away (after $\sim Q$ periods of ω_0),

$$x(t) = Ce^{i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

If the driving force had been $F_0 e^{-i\omega t}$, we would have found

$$x(t) = Ce^{-i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 - 2i\beta\omega + \omega_0^2}$$

Linear superposition lets us average these two solutions to get the response to real driving force $F_0 \cos(\omega t)$.

For driving force $F_0 \cos(\omega t) = \frac{F_0}{2}(e^{i\omega t} + e^{-i\omega t})$, we get (after transients die out)

$$x(t) = \frac{F_0/2}{-\omega^2 + 2i\beta\omega + \omega_0^2}\ e^{i\omega t} + \frac{F_0/2}{-\omega^2 - 2i\beta\omega + \omega_0^2}\ e^{-i\omega t}$$

which is the same as

$$x(t) = \mathrm{Re}\left(\frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}\ e^{i\omega t}\right)$$

which after some algebra is

$$x(t) = A\cos(\omega t - \delta)$$

with

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

[Mathematica and physical demos]

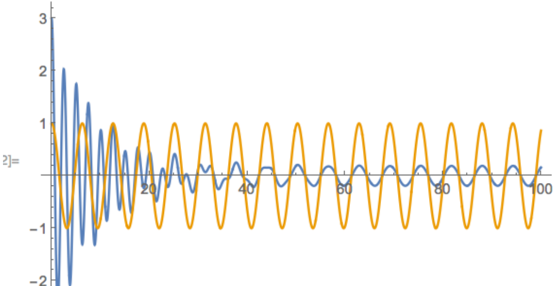
For driving force $F_0 \cos(\omega t)$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + A\cos(\omega t - \delta)$$

with $\Omega = \sqrt{\beta^2 - \omega_0^2} = i\omega_1$

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The important point to remember (for the usual "underdamped" case) is that the transient response rings at $\omega_1 \approx \omega_0$, which is close to the natural frequency, and decays away at rate β. But the long-term response is at the driving frequency ω, with an amplitude and phase that depend on $\omega - \omega_0$.



```
ClearAll["Global`*"];
omega0 = 2.5; omega = 1.0; beta = 0.1; fampl = 1.0; fphi = 0.0; x0 = 3.0;
v0 = 0.0; tmax = 100.0;
soln = NDSolveValue[{x''[t] + 2 beta x'[t] + omega0^2 x[t] == fampl Cos[omega t + fphi],
    x'[0] == v0, x[0] == x0}, x[t], {t, 0, tmax}];
Plot[{soln, fampl Cos[omega t + fphi]}, {t, 0, tmax}, PlotRange → All]
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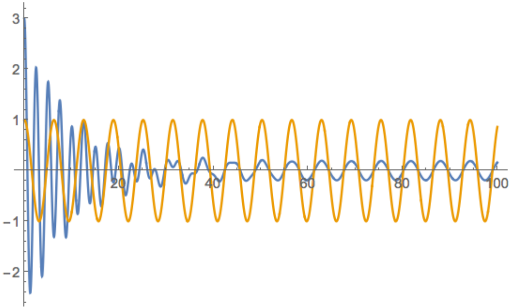
752]=



Simpler example (useful for homework) on next page.

In[5]:=
```
ClearAll["Global`*"];
g = 9.8;
l = 1.0;
soln = NDSolveValue[
  {θ''[t] + (g / l) Sin[θ[t]] == 0, θ'[0] == 0,
   θ[0] == 0.1}, θ[t], {t, 0, 10}]
```

Out[8]=
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InterpolatingFunction[ Domain: {{0., 10.}} Output: scalar ][t]
```

In[9]:=
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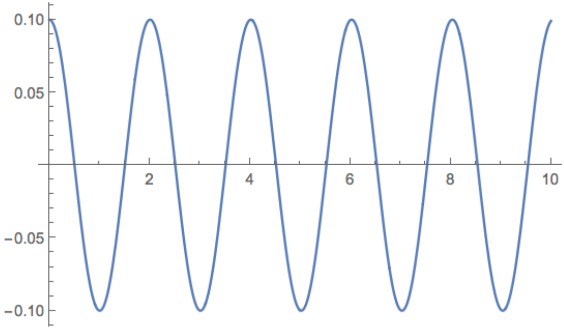
Plot[soln, {t, 0, 10}]
```

Out[9]=



Let's go back to the complex-number driving force

For driving force $F_0 e^{i\omega t}$, we found

$$x(t) = e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t}) + Ce^{i\omega t}$$

Once the transients have died away (after $\sim Q$ periods of ω_0),

$$x(t) = Ce^{i\omega t}$$

with

$$C = \frac{F_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

Now suppose you have a more complicated driving force:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_a\, e^{i\omega_a t} + F_b\, e^{i\omega_b t}$$

Since $\mathcal{D}$ is linear,

$$\mathcal{D}\left[\frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2}\right] = F_a e^{i\omega_a t}$$

$$\mathcal{D}\left[\frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2}\right] = F_b e^{i\omega_b t}$$

$$\mathcal{D}\left[\frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2} + \frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2}\right] = F_a e^{i\omega_a t} + F_b e^{i\omega_b t}$$

So the general solution is

$$x(t) = \frac{F_a\, e^{i\omega_a t}}{-\omega_a^2 + 2i\beta\omega_a + \omega_0^2} + \frac{F_b\, e^{i\omega_b t}}{-\omega_b^2 + 2i\beta\omega_b + \omega_0^2} + e^{-\beta t}(Ae^{+\Omega t} + Be^{-\Omega t})$$

where again the transient terms are irrelevant for $t \gg 1/\beta$.

Now consider the more general case

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

and suppose we're able to write

$$f(t) = \sum_n F_n\, e^{i\omega_n t}$$

Then it's clear that the solution would be

$$x(t) = (\text{transient}) + \sum_n \frac{F_n\, e^{i\omega_n t}}{-\omega_n^2 + 2i\beta\omega_n + \omega_0^2}$$

If $f(t)$ is periodic (period $T \equiv 2\pi/\omega$), then Prof. Fourier tells us

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega t}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t}$$

$$\frac{1}{T}\int_{-T/2}^{+T/2} f(t)\, e^{-im\omega t}\,\mathrm{d}t = \sum_n \frac{F_n}{T}\int_{-T/2}^{+T/2} \mathrm{d}t\, e^{i(n-m)\omega t} = \sum_n F_n \delta_{mn} = F_m$$

So the Fourier coefficient F_m is

$$F_m = \frac{1}{T}\int_{-T/2}^{+T/2} f(t) e^{-im\omega t}\,\mathrm{d}t$$

Note: for $f(t)$ real, $F_{-m} = F_m^*$, i.e. the negative-frequency coefficients are the complex conjugates of the corresponding positive-frequency coefficients.

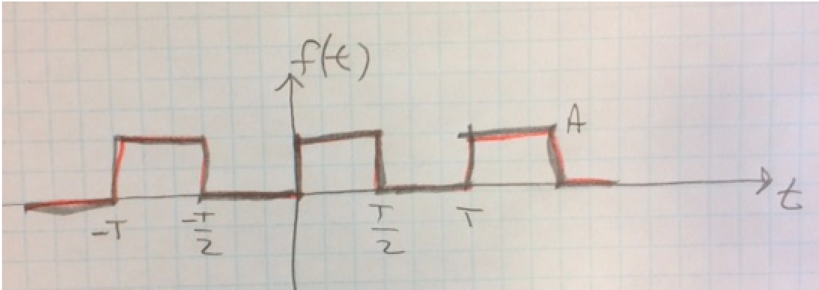
$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t}$$

with Fourier coefficient F_n given by

$$F_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-in\omega t} \, \mathrm{d}t$$

Exercise: use this complex-number Fourier formalism to find the Fourier series for a square wave $f(t)$ of period $T = 2\pi/\omega$, with

$f(t) = 0$ for $-T/2 < t < 0$
$f(t) = A$ for $0 < t < T/2$



Square wave: $f(t) = \begin{array}{ll} 0 \text{ if } & -T/2 < t < 0 \\ A \text{ if } & 0 < t < T/2 \end{array}$ $\quad T = \frac{2\pi}{\omega}$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega t}dt = \frac{A}{T}\int_0^{T/2} e^{-in\omega t}dt = \frac{A}{-in\omega T}\left[e^{-in\omega T/2} - 1\right]$$

$$= \frac{A}{-in2\pi}\left(e^{-in\pi} - 1\right) = \frac{A}{in2\pi}\left(1 - (-1)^n\right)$$

$F_n = 0$ for even $n \neq 0$ $\qquad F_n = \frac{A}{in\pi}$ for odd n

$$F_0 = \frac{A}{T}\int_0^{T/2} dt = \frac{A}{2} = \langle f(t) \rangle$$

$$f(t) = \frac{A}{2} + \sum_{n=1,3,5,\ldots} \frac{A}{in\pi}\left(\underbrace{e^{+in\omega t} - e^{-in\omega t}}_{2i\sin(n\omega t)}\right)$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left(\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \cdots\right)$$

699]:=
```
f[t_] := 1/2 + (2/Pi) (Sin[t] + Sin[3 t]/3 + Sin[5 t]/5);
Plot[f[t], {t, -10, 10}]
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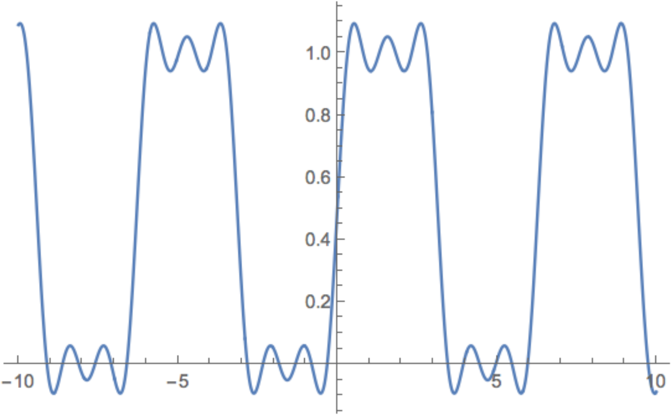
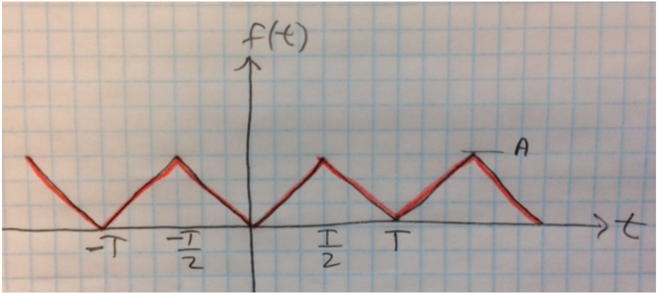
700]=



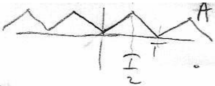
$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{in\omega t} \qquad\qquad F_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(t) e^{-in\omega t} \, \mathrm{d}t$$

Exercise: use this complex-number Fourier formalism to find the Fourier series for a triangle wave $f(t)$ of period $T = 2\pi/\omega$, with

$f(t) = -2At/T$ for $-T/2 < t < 0$
$f(t) = 2At/T$ for $0 < t < T/2$



$$\text{hint}: \quad \int t e^{-in\omega t} \mathrm{d}t \; = \; \frac{1 + in\omega t}{(n\omega)^2} \, e^{-in\omega t}$$



$$f(t) = \begin{cases} \dfrac{2At}{T} & 0 < t < \frac{T}{2} \\ -\dfrac{2At}{T} & -\frac{T}{2} < t < 0 \end{cases}$$

$$T = \frac{2\pi}{\omega}$$

$$\omega T = 2\pi$$

$$\omega T/2 = \pi$$

$$F_m = \frac{2A}{T^2}\left(\int_0^{T/2} t e^{-im\omega t}\,dt - \int_{-T/2}^{0} t e^{-im\omega t}\,dt\right)$$

$$= \frac{2A}{T^2}\left(\left[\frac{e^{-im\omega t}(1+im\omega t)}{(m\omega)^2}\right]_0^{T/2} - \left[\frac{e^{-im\omega t}(1+im\omega t)}{(m\omega)^2}\right]_{-T/2}^{0}\right)$$

$$= \frac{2A}{(m\omega T)^2}\left(e^{-im\pi}(1+im\pi) - 1 - 1 + e^{im\pi}(1-im\pi)\right)$$

$$= \frac{2A}{(m2\pi)^2}\left((-1)^m \left(1+\cancel{im\pi}\right) - 2 + (-1)^m \left(1-\cancel{im\pi}\right)\right)$$

$$= \frac{A}{(m\pi)^2}\left((-1)^m - 1\right) \qquad = 0 \text{ for even } m \neq 0$$

$$\qquad\qquad = \frac{-2A}{(m\pi)^2} \text{ for odd } m$$

$$F_0 = \frac{2A}{T^2}\left(2\int_0^{T/2} t\,dt\right) = \frac{4A}{T^2}\left[\frac{1}{2}\left(\frac{T}{2}\right)^2\right] = \frac{A}{2} \quad \checkmark$$

$$f(t) = \frac{A}{2} - \sum_{n=1,3,5,\ldots} \frac{2A}{(n\pi)^2}\left(\underbrace{e^{in\omega t} + e^{-in\omega t}}_{2\cos(n\omega t)}\right)$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\left(\cos(\omega t) + \frac{1}{9}\cos(3\omega t) + \frac{1}{25}\cos(5\omega t)\right)$$

```
f[t_] := 1 / 2 - (4 / Pi ^ 2) (Cos[t] + Cos[3 t] / 9 + Cos[5 t] / 25);
Plot[f[t], {t, -10, 10}]
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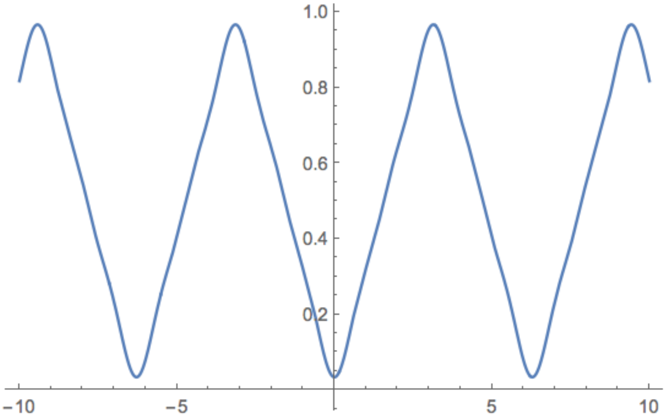
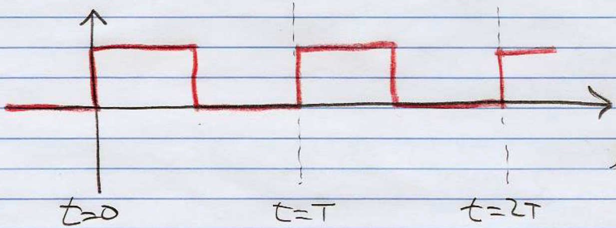


Example: periodic square wave



$$f(t) = \begin{cases} f_A & 0 < t < \frac{T}{2} \\ 0 & -\frac{T}{2} < t < 0 \end{cases}$$

let $\omega = \frac{2\pi}{T}$

$$F_m = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)\, e^{-im\omega t} = \frac{f_A}{T}\int_0^{T/2} e^{-im\left(\frac{2\pi}{T}\right)t}\, dt$$

$$= \frac{f_A}{T}\left(\frac{T}{-2\pi i m}\right)\left[e^{-im\left(\frac{2\pi}{T}\right)t}\right]_0^{T/2}$$

$$= -\frac{f_A}{2\pi i m}\left(\left(e^{-i\pi}\right)^m - 1\right) = \frac{f_A}{2\pi i m}\left(1-(-1)^m\right)$$

For m odd, $F_m = \dfrac{2f_A}{2\pi i m} = \dfrac{f_A}{i\pi m}$

For even $m \neq 0$, $F_m = 0$.

For $m=0$, $F_0 = \dfrac{f_A}{T}\displaystyle\int_0^{T/2} dt = \dfrac{f_A}{T}\left(\dfrac{T}{2}\right) = \boxed{\dfrac{f_A}{2} = F_0}$

(F_0 is just the average $\langle f(t) \rangle$)

So $x(t) = (\text{transient}) + \displaystyle\sum_n \frac{F_n e^{in\omega t}}{(\omega_0^2 - n^2\omega^2) + 2in\beta\omega}$

Dropping uninteresting transient,

$$x(t) = \sum_n \frac{\left(F_n e^{in\omega t}\right)\left((\omega_0^2 - n^2\omega^2) - 2in\beta\omega\right)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

$$x(t) = \frac{F_0}{\omega_0^2} + \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(e^{in\omega t} - e^{-in\omega t}\right)\left(\omega_0^2 - n^2\omega^2\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

$$+ \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(e^{in\omega t} + e^{-in\omega t}\right)\left(-2in\beta\omega\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(2i\sin(n\omega t)\right)\left(\omega_0^2 - n^2\omega^2\right)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

$$+ \sum_{\substack{n= \\ 1,3,5,\ldots}} \frac{\left(\frac{f_A}{i\pi n}\right)\left(2\cos(n\omega t)\right)\left(-2in\beta\omega\right)}{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum \frac{\left(\frac{2f_A}{\pi n}\right)\sin(n\omega t)\left(\omega_0^2 - n^2\omega^2\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

$$+ \sum \frac{\left(\frac{2f_A}{\pi n}\right)\cos(n\omega t)\left(-2n\beta\omega\right)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\beta\omega)^2}$$

Now let $C_n = \dfrac{\omega_0^2 - n^2\omega^2}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}} \equiv \cos(\delta_n)$

$$S_n = \frac{-2n\beta\omega}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}} \equiv \sin(\delta_n)$$

notice that $C_n^2 + S_n^2 = 1$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{n=1,3,5,\ldots} \sin(n\omega t)\cos(\delta_n)\frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$+ \sum_{n=1,3,5,\ldots} \cos(n\omega t)\sin(-\delta_n)\frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$x(t) = \frac{f_A}{2\omega_0^2} + \sum_{n=1,3,5,\dots} A_n \sin(n\omega t - \delta_n)$$

where $$A_n = \frac{2f_A}{\pi n\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2n\beta\omega)^2}}$$

$$\sin(n\omega t - \delta_n) = \sin(\omega t)\cos(\delta) + \cos(\omega t)\sin(-\delta)$$

$$\delta_n = \arctan\left(\frac{2n\beta\omega}{\omega_0^2 - n^2\omega^2}\right)$$

Notice that the answer came out entirely real, even though we used complex exponentials. Also notice that this looks just like the result from the book using sines and cosines:

$x_n(t) = A_n \cos(n\omega t - \delta_n)$ $\quad A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}}$ $\quad$ same δ_n

and $2f_A/(\pi n)$ is just f_n. ($\sin$ vs. $\cos$ depends on chosen time offset of square wave.)

Physics 351 — Wednesday, January 24, 2018

- Homework #1 due on Friday. Start reading ch7 for Monday.
- I'll hand out HW #2 in class on Friday, and I'll put the PDF online some time tomorrow.
- You can do the Mathematica extra credit any time you like (if at all), but the earlier you do it, the more you'll be able to make use of Mathematica to reduce tedious algebra in your own homework.
- Homework help sessions start Jan 24–25 (Wed/Thu).
- Bill will be in DRL 3N6 on Wednesdays 4pm–7pm
- Grace will be in DRL 2C2 on Thursdays 5:30pm-8:30pm
- We'll spend the rest of this week on Ch 5–6.