Physics 351 — Wednesday, January 31, 2018

- HW2 due Friday. Next weekend you'll read the rest of Ch 7.
- HW help: Bill is in DRL 3N6 Wednesdays 4pm–7pm.
  Grace is in DRL 2C2 Thursdays 5:30pm-8:30pm.

Before class: solve the hanging-chain problem by solving for $x(y)$ instead of solving for $y(x)$. Write

$$U \;=\; \mu g \int y\,\mathrm{d}s \;=\; \mu g \int y\sqrt{1+x'^2}\,\mathrm{d}y$$

Now we have $\int \mathrm{d}y\, f(y, x, x')$, so the Euler-Lagrange equation becomes

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial f}{\partial x'}$$

and we can use the "momentum conservation" trick.

At the end of Monday's class, we solved the "hanging chain" (catenary) problem in the most direct way, by finding $y(x)$.

$$U = \mu g \int y \, \mathrm{d}s = \mu g \int y \sqrt{1 + y'^2} \, \mathrm{d}x$$

We wrote down

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'}$$

and after some messy algebra and a lot of cancellation we got

$$1 + (y')^2 - y \, y'' = 0$$

Hanging chain – the first way (from Monday)

$$f(x,y,y') = y\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = \sqrt{1+y'^2} \quad \Bigg\} \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)$$

$$= \frac{y'y'}{\sqrt{1+y'^2}} + \frac{yy''}{\sqrt{1+y'^2}} - \frac{yy'\;y'y''}{(1+y'^2)^{3/2}}$$

$$\sqrt{1+y'^2} = \frac{y'^2+yy''}{\sqrt{1+y'^2}} - \frac{y(y'^2)y''}{(1+y'^2)^{3/2}}$$

$$(1+y'^2)^2 = (y'^2+yy'')(1+y'^2) - y(y'^2)y''$$

$$1+2y'^2+\underline{y'^4} = y'^2+\underline{y'^4}+yy''+\underline{yy''y'^2}-\underline{yy''y'^2}$$

$$1+y'^2-yy''=0$$

$$1 + y'^2 - yy'' = 0$$

If you happen to guess $y = A\cosh\left(\frac{x-B}{A}\right)$
then

$$y' = \sinh\left(\frac{x-B}{A}\right) \rightarrow y'^2 = \sinh^2\left(\frac{x-B}{A}\right)$$

$$y'' = \left(\frac{1}{A}\right)\cosh\left(\frac{x-B}{A}\right) \rightarrow yy'' = \cosh^2\left(\frac{x-B}{A}\right)$$

$$1 + y'^2 - yy'' = 1 + \sinh^2\left(\frac{x-B}{A}\right) - \cosh^2\left(\frac{x-B}{A}\right) = 1 - 1 = 0$$

which satisfies $1 + y'^2 - yy'' = 0$ ✓

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Plot[Cosh[x], {x, -1, 1}, LabelStyle → 18]
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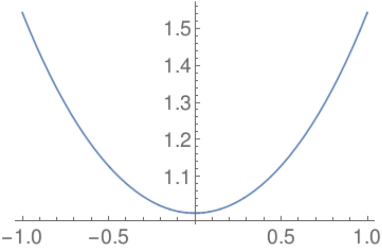

A second way to approach this same problem is to solve for $x(y)$ instead of solving for $y(x)$.

$$U \;=\; \mu g \int y \,\mathrm{d}s \;=\; \mu g \int y\sqrt{1+x'^2}\,\mathrm{d}y$$

Now we have $\int \mathrm{d}y f(y, x, x')$, so the Euler-Lagrange equation becomes

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial f}{\partial x'}$$

Now we can use the "momentum conservation" trick, because $f(y, x, x')$ is independent of x.

$$f(y, x, x') = y\sqrt{1+x'^2}$$

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'} \Rightarrow \frac{\partial f}{\partial x} = 0 = \frac{d}{dy}\left(\frac{y\,x'}{\sqrt{1+x'^2}}\right)$$

$$\Rightarrow \frac{y\,x'}{\sqrt{1+x'^2}} = c$$

$$y^2x'^2 = c^2(1+x'^2) \Rightarrow (y^2-c^2)\,x'^2 = c^2$$

$$x' = \frac{c}{\sqrt{y^2-c^2}}$$ which can be integrated to get

$$x = c\cosh^{-1}\left(\frac{y}{c}\right) + b$$

$$\cosh\left(\frac{x-b}{c}\right) = \frac{y}{c} \Rightarrow y = c\cosh\left(\frac{x-b}{c}\right)$$

So the "conserved momentum" trick — $f(y, x, x')$ indep. of x, which is analogous to $\mathcal{L}(t, x, \dot{x})$ indep. of x — gave us a first-order ODE (compare with $1 + y'^2 - yy'' = 0$), which we could integrate instead of having to guess a solution.

In[8]:=
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ClearAll["Global`*"];
D[c ArcCosh[y / c] + b, y]
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Out[9]=
$$\frac{1}{\sqrt{-1+\frac{y}{c}}\,\sqrt{1+\frac{y}{c}}}$$

There is a second trick for getting a first-order ODE from the E-L equation, which works if $f(x,y,y') = f(y,y')$ is independent of x. This trick turns out to be analogous to the total energy being conserved, if $\mathcal{L}(t,x,\dot{x})$ is independent of t.

<u>Claim</u>:
If $f(x,y,y') = f(y,y')$ i.e. $f(x,y,y')$ is independent of x, then any function $y(x)$ that extremizes $\int \mathrm{d}x\, f(y,y')$ satisfies

$$h \;\equiv\; y' \frac{\partial f}{\partial y'} - f(y,y') \;=\; \text{constant}$$

In the context of mechanics, h will be the "Hamiltonian," which equals the (constant) total energy if $\mathcal{L}(t,x,\dot{x})$ is independent of t.

Given: $y(x)$ extremizes $\int \mathrm{d}x\, f(y, y')$

Claim:

$$h \;\equiv\; y' \frac{\partial f}{\partial y'} - f(y, y') \;=\; \text{constant}$$

Proof:

$$\frac{\mathrm{d}h}{\mathrm{d}x} \;=\; y'' \frac{\partial f}{\partial y'} + y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial f}{\partial y'} \frac{\mathrm{d}y'}{\mathrm{d}x}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} \;=\; y'' \frac{\partial f}{\partial y'} + y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - y' \frac{\partial f}{\partial y} - y'' \frac{\partial f}{\partial y'}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} \;=\; y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - y' \frac{\partial f}{\partial y}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} \;=\; y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right) \;=\; 0$$

because $y(x)$ must satisfy the Euler-Lagrange equation. QED.

Back to our hanging-chain problem. $f(y, y') = y\sqrt{1 + y'^2}$.

$$h \;\equiv\; y' \frac{\partial f}{\partial y'} - f(y, y') \;=\; \text{constant}$$

Back to our hanging-chain problem. $f(y, y') = y\sqrt{1+y'^2}$.

$$h \equiv y' \frac{\partial f}{\partial y'} - f(y, y') = \text{constant}$$

$$f(y,y') = y\sqrt{1+y'^2}$$

$$h = y'\frac{\partial f}{\partial y'} - f = y'\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - y\sqrt{1+y'^2} = h$$

$$yy'^2 - y(1+y'^2) = h\sqrt{1+y'^2}$$

$$yy'^2 - y - yy'^2 = h\sqrt{1+y'^2}$$

$$-y = h\sqrt{1+y'^2} \Rightarrow \left(\frac{y}{h}\right)^2 = 1+y'^2$$

which you could solve e.g. by guessing

$$y = h\cosh\left(\frac{x-B}{h}\right)$$

$$y' = \sinh\left(\frac{x-B}{h}\right) \rightarrow 1+y'^2 = 1+\sinh^2\left(\frac{x-B}{A}\right) = \cosh^2\left(\frac{x-B}{A}\right)$$

$$1+y'^2 = \left(\frac{y}{h}\right)^2 \checkmark$$

If you formally study the Calculus of Variations, the "momentum" trick is called the "first integral" of the E-L equation, and the "energy" trick is called the "Baltrami identity."

I mention them for two reasons: Most importantly, because we will see mechanical analogues of these tricks in the next two weeks. In the first case, "ignorable coordinates" (coordinates not appearing in the Lagrangian) lead to "conserved (generalized) momenta." In the second case, conservation of energy is expressed (in mechanics) by writing down a Hamiltonian function.

The second reason is that first-order ODEs are generally easier to solve than second-order ODEs, so these two tricks can save effort when using the E-L equation for optimization problems.

If you're a physicist, you'll often find that the easiest way to remember a given math result is to remember the analogous physics problem for which it is useful!

Now, a moment you've been waiting 2.5 weeks for!

In last weekend's reading (the non-asterisk parts of Ch7), you saw that the trajectory of a particle moving in potential $U(x)$ follows the "path of least action," i.e. it follows the path $x(t)$ for which the "action" $S[x(t)]$ is stationary:

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\, \mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\, \mathrm{d}t$$

Feynman points out (XC part of next weekend's reading)

`http://www.feynmanlectures.caltech.edu/II_19.html`

that to be precise, one should really call the Lagrangian approach "the principle of stationary Hamilton's first principal function." But most people say, more concisely, "the principle of least action."

Lagrangian mechanics: Use the Euler-Lagrange equation to find the trajectory $x(t)$ for which the "action" $S[x]$ is stationary.

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\,\mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\,\mathrm{d}t$$

Let's give it a try for a particle of mass m dropped vertically from a short distance x above Earth's surface. For notational simplicity, let the x axis point **vertically upward**. (I should draw this.)

First write down $\mathcal{L}(t, x, \dot{x})$. (Try it!)

Then write the E-L equation: (which variables are which here?)

Lagrangian mechanics: Use the Euler-Lagrange equation to find the trajectory $x(t)$ for which the "action" $S[x]$ is stationary.

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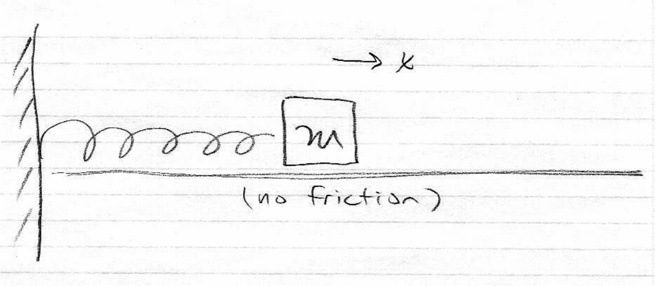
First write down $\mathcal{L}(t, x, \dot{x})$. (Try it!)

Then write the E-L equation: (which variables are which here?)

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

Then turn the crank and see what EOM pops out. (Try it!)

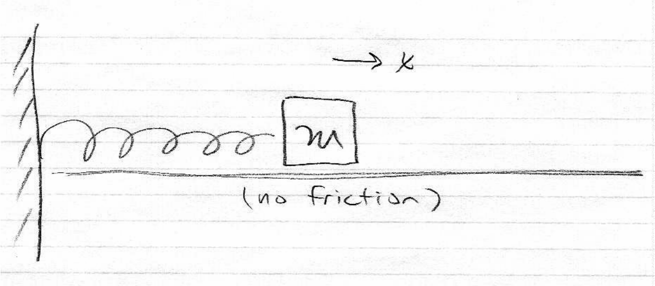


OK, how about a block of mass m moving horizontally on a frictionless table, under the influence of Hooke's-Law potential

$$U = \frac{1}{2}kx^2$$

so $x = 0$ when spring is at its equilibrium length.

Try writing down $\mathcal{L}$, then using $E - L$ equations to find EOM.



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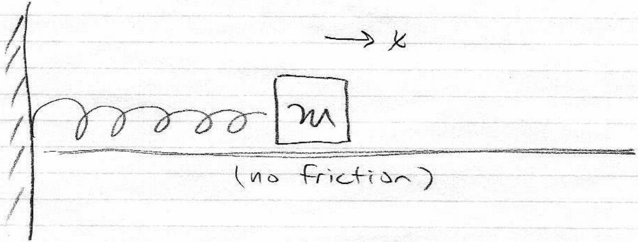
What EOM do we get for a general 1D potential $U(x)$?



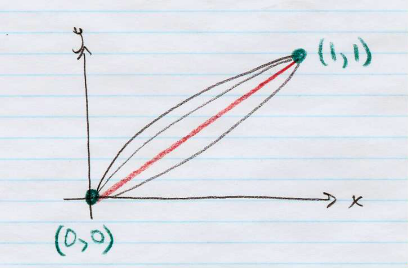
$$T = \frac{1}{2} m \dot{x}^2 \qquad U = \frac{1}{2} k x^2$$

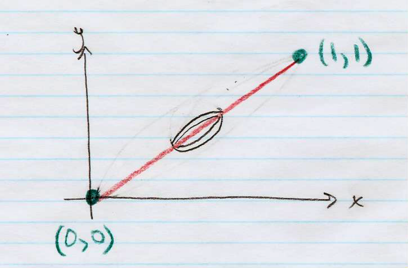
$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \Rightarrow -kx = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x}$$

$$\boxed{m\ddot{x} = -kx}$$ (no surprise!)

When you use the E-L equation to optimize $\int f(x, y, y')\, \mathrm{d}x$, you may be thinking globally, but the E-L equation is acting locally.





The $y(x)$ that makes path length $\int \sqrt{1 + y'^2}\, \mathrm{d}x$ optimal from $(0, 0)$ to $(1, 1)$ will also make the path length optimal from $(0.4, 0.4)$ to $(0.6, 0.6)$ or from $(0.49, 0.49)$ to $(0.51, 0.51)$.

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'}$$

is imposing a **local** requirement on $y(x)$. It's acting on each little segment of the curve separately.

Similarly,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

is enforcing $\vec{F} = m\vec{a}$ locally at each step in time. But the effect is that the overall path optimizes the action $S = \int \mathcal{L} \, \mathrm{d}t$.

Lagrangian mechanics: Use the Euler-Lagrange equation to find the trajectory $x(t)$ for which the "action" $S[x]$ is stationary.

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\,\mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\,\mathrm{d}t$$

As most of you quoted last weekend:

(1) Write down the K.E. and P.E. and hence the Lagrangian $\mathcal{L} = T - U$ using any convenient **inertial** reference frame.

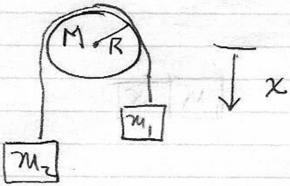
(2) Choose a convenient set of n generalized coordinates q_i and solve for original coords (from step 1) in terms of $q_1 \ldots q_n$.

(3) Rewrite $\mathcal{L}$ in terms of q_i and $\dot{q}_i$.

(4) Write down the n Lagrange equations.

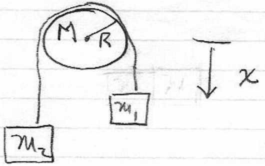
$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Let's work through several examples together, starting from some basic one-variable cases, then becoming more complicated.



Write $L = T - U$ for Atwood machine in which pulley (mass M, radius R) has non-negligible inertia. Take pulley to be a solid cylinder.

After writing down $\mathcal{L}$, use Lagrange eqns to write EOM for $x(t)$, i.e. the expression for $\ddot{x}$.



$$T = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{4}M\dot{x}^2 = \frac{1}{2}\left(m_1+m_2+\frac{M}{2}\right)\dot{x}^2$$

$$U = (m_2-m_1)gx$$

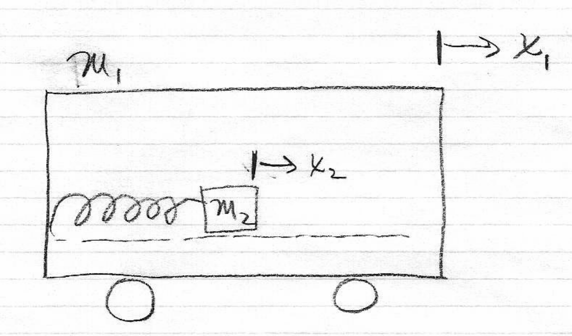
$$L = T-U = \frac{1}{2}\left(m_1+m_2+\frac{M}{2}\right)\dot{x}^2 + (m_1-m_2)gx$$

$$\frac{\partial L}{\partial x} = (m_1-m_2)g = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m_1+m_2+\frac{M}{2}\right)\ddot{x}$$

$$\boxed{\ddot{x} = \frac{(m_1-m_2)}{m_1+m_2+\frac{M}{2}}\,g}$$

Reading question: "Does the Lagrangian method still work if one chooses generalized coordinates relative to a non-inertial reference frame? If so, is there some precaution one needs to take in writing down the Lagrangian?"

Yes: Lagrange's equations are true for any choice of generalized coordinates, even if they are relative to a non-inertial frame. One just has to be careful to write the Lagrangian L=T-U in an inertial frame.



A cart of mass m_1 rolls horizontally without friction. The cart's position is x_1. Inside the cart, a mass m_2 is attached to the wall of the cart with a spring (constant k). The position of m_2 w.r.t. the spring's relaxed position is x_2. So x_2 is w.r.t. the cart, not w.r.t. the ground. Write $\mathcal{L}(t, x_1, \dot{x}_1, x_2, \dot{x}_2)$.

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 - \frac{1}{2} k x_2^2$$

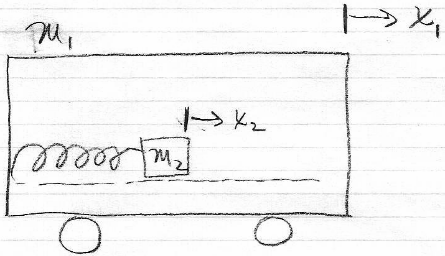
$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = \frac{d}{dt}\left[m_1 \dot{x}_1 + m_2 (\dot{x}_1 + \dot{x}_2) \right]$$

$$0 = (m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = -k x_2 = \frac{d}{dt}\left[m_2 (\dot{x}_1 + \dot{x}_2) \right]$$

$$-k x_2 = m_2 \ddot{x}_1 + m_2 \ddot{x}_2$$

By the way, notice that x_1 is an "ignorable" (a.k.a. "cyclic") coordinate, i.e. $\partial \mathcal{L} / \partial x_1 = 0$. The corresponding conserved quantity is the momentum of the CM, $m_1 \dot{x}_1 + m_2(\dot{x}_1 + \dot{x}_2)$.

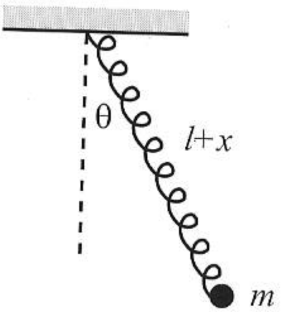


$$x_{cm} = \frac{m_1 x_1 + m_2(x_1+x_2)}{m_1+m_2}$$

$$(m_1+m_2)\,x_{cm} = m_1 x_1 + m_2(x_1+x_2)$$
$$= (m_1+m_2)\,x_1 + m_2 x_2$$

$$(m_1+m_2)\,\ddot{x}_{cm} = (m_1+m_2)\,\ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

$$\Rightarrow \ddot{x}_{cm} = 0$$



Consider a pendulum made of a spring with a mass m on the end. The spring is arranged to lie in a straight line (e.g. by wrapping the spring around a massless rod). The equilibrium length of the spring is ℓ. Let the spring have length $\ell + x(t)$, and let its angle w.r.t. vertical be $\theta(t)$. Assuming the motion takes place in a vertical plane, write Lagrangian and find EOM for x and θ.

$$T = \frac{1}{2} m (l+x)^2 \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2 - mg(x+l)\cos\theta$$

$$L = \frac{1}{2} m (l+x)^2 \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2 + mg(x+l)\cos\theta - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = mg\cos\theta - kx + m(l+x)\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \Rightarrow \boxed{m\ddot{x} = mg\cos\theta - kx + m(l+x)\dot{\theta}^2}$$

$$\frac{\partial L}{\partial \theta} = -mg(x+l)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m(l+x)^2\dot{\theta}\right) = m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta}$$

$$\Rightarrow m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} = -mg(x+l)\sin\theta$$

$$\Rightarrow \boxed{(l+x)\ddot{\theta} = -g\sin\theta - 2\dot{x}\dot{\theta}}$$

(Notice "$\omega^2 r$" centripetal term and "$2\omega\dot{r}$" Coriolis term.)

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