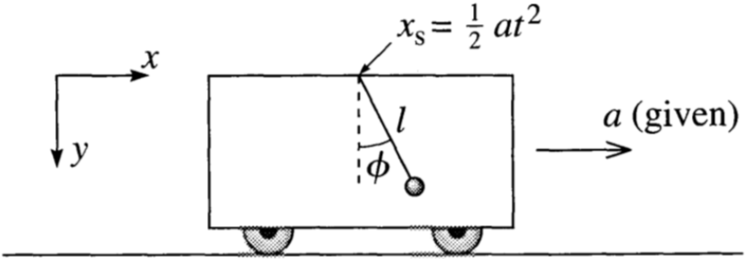
Physics 351 — Wednesday, February 7, 2018

- HW3 due Friday. You finished reading ch7 last weekend. You'll read ch8 (Kepler problem) this weekend.
- HW help: Bill is in DRL 3N6 Wednesdays 4pm–7pm. Grace is in DRL 2C2 Thursdays 5:30pm-8:30pm.

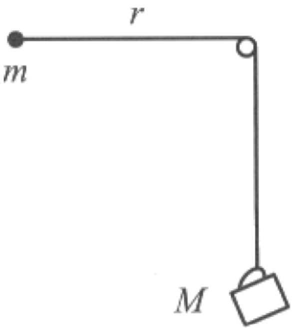


(7.30) A pendulum is suspended inside a railroad car that is forced to accelerate at constant acceleration a.

(a) Write down $\mathcal{L}$ and find EOM for ϕ.

One interesting feature of this problem is that it is non-linear and cannot be solved analytically. In fact, at very large amplitude it behaves chaotically: something we will briefly explore when you read chapter 12 toward the end of the semester. (For now this is just a digression.)

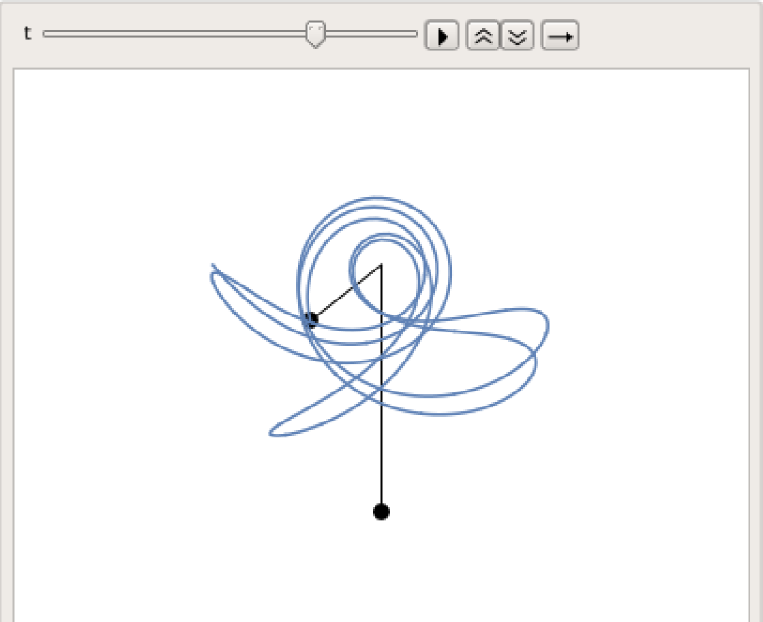
A coffee cup of mass M is connected to a mass m by a string. The coffee cup hangs over a frictionless pulley of negligible size, and the mass m is initially held with the string horizontal, as shown in the figure. The mass m is then released. (a) Find the EOM for r (the length of string between m and the pulley) and θ (the angle that the string to m makes with the horizontal). Assume that m somehow doesn't run into the string holding the cup up. The coffee cup will initially fall, but it turns out that it will reach a lowest point and then rise back up. (b) Use Mathematica (or similar) to determine numerically the ratio of the r at this lowest point to the r at the start, as a function of the value of m/M. (To check your computation, a value of $m/M = 1/10$ yields a ratio of about 0.208.)



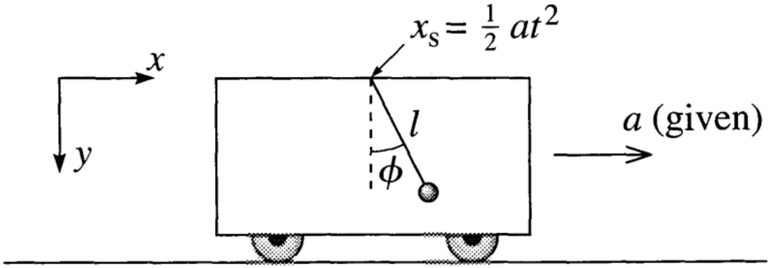
Crucial hint: the two coupled EOM can't be solved analytically. Use `NDSolveValue` then `FindMinimum` in Mathematica.

Non-linear behavior is evident at large amplitude!! (Graph by 2015 student Noah Rubin — he did this just for fun.)

Out[10]=



This problem will reappear in the text of Taylor's Ch9 ("mechanics in non-inertial frames"), so let's work through it by writing the Lagrangian w.r.t. an inertial frame.



(7.30) A pendulum is suspended inside a railroad car that is forced to accelerate at constant acceleration a.

(a) Write down $\mathcal{L}$ and find EOM for ϕ.

(b) Let $\tan\beta \equiv a/g$, so $g = \sqrt{g^2+a^2}\cos\beta$, $a = \sqrt{g^2+a^2}\sin\beta$. Simplify using $\sin(\phi+\beta) = \cos\beta\sin\phi + \sin\beta\cos\phi$.

(c) Find equilibrium angle ϕ_0. Use EOM to show $\phi = \phi_0$ is stable. Find frequency of small oscillations about ϕ_0.

$$U = -mgy_m = -mgl\cos\phi$$

$$x_m = \frac{1}{2}at^2 + l\sin\phi \qquad y_m = l\cos\phi$$

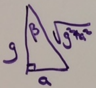
$$\dot{x}_m = at + l\dot{\phi}\cos\phi \qquad \dot{y}_m = -l\dot{\phi}\sin\phi$$

$$\dot{x}_m^2 + \dot{y}_m^2 = (at)^2 + (l\dot{\phi})^2(\cos^2\phi + \sin^2\phi) + 2atl\dot{\phi}\cos\phi$$

$$\mathcal{L} = \frac{1}{2}ma^2t^2 + \frac{1}{2}ml^2\dot{\phi}^2 + matl\dot{\phi}\cos\phi + mgl\cos\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -(\cancel{matl\dot{\phi}} + mgl)\sin\phi = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(ml^2\dot{\phi} + matl\cos\phi\right) = ml^2\ddot{\phi} + mal\cos\phi - \cancel{matl\dot{\phi}\sin\phi}$$

$$ml^2\ddot{\phi} + mal\cos\phi + mgl\sin\phi = 0 \Rightarrow \ddot{\phi} + \frac{a}{l}\cos\phi + \frac{g}{l}\sin\phi = 0$$



$$g = (\cos\beta)\sqrt{g^2+a^2}$$
$$a = (\sin\beta)\sqrt{g^2+a^2}$$

$$\ddot{\phi} + \frac{\left(\sqrt{g^2+a^2}\right)}{l}\left(\sin\beta\cos\phi + \cos\beta\sin\phi\right) = 0$$

$$\ddot{\phi} + \frac{\sqrt{g^2+a^2}}{l}\sin(\phi+\beta) = 0$$

let $\varepsilon = \phi + \beta \Rightarrow \ddot{\varepsilon} = \ddot{\phi} \Rightarrow \ddot{\varepsilon} + \frac{\sqrt{g^2+a^2}}{l}\sin\varepsilon = 0$

for $\varepsilon \ll 1$ we get $\ddot{\varepsilon} = -\frac{\sqrt{g^2+a^2}}{l}\varepsilon$

$$x_m = \frac{1}{2}at^2 + l\sin\phi \qquad y_m = l(1-\cos\phi)$$

$$\dot{x}_m = at + l\cos\phi\,\dot{\phi} \qquad \dot{y}_m = l\sin\phi\,\dot{\phi}$$

$$\dot{x}_m^2 + \dot{y}_m^2 = a^2t^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2atl\cos\phi\,\dot{\phi} + l^2\sin^2\phi\,\dot{\phi}^2$$

$$= a^2t^2 + l^2\dot{\phi}^2 + 2atl\cos\phi\,\dot{\phi}$$

$$\boxed{U = mgl(1-\cos\phi)}$$

$$T = \frac{1}{2}m\left(a^2t^2 + l^2\dot{\phi}^2\right) + matl\cos\phi\,\dot{\phi}$$

$$L = \frac{m}{2}\left(a^2t^2 + l^2\dot{\phi}^2\right) + matl\cos\phi\,\dot{\phi} + mgl(\cos\phi - 1)$$

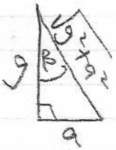
$$\frac{\partial L}{\partial \phi} = -matl\sin\phi\,\dot{\phi} - mgl\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(ml^2\dot{\phi} + matl\cos\phi\right) = ml^2\ddot{\phi} + mal\cos\phi - matl\sin\phi\,\dot{\phi}$$

$$ml^2\ddot{\phi} + mal\cos\phi = -mgl\sin\phi \Rightarrow l\ddot{\phi} = -g\sin\phi - a\cos\phi$$

$$ml^2\ddot{\phi} + mal\cos\phi = -mgl\sin\phi \Rightarrow l\ddot{\phi} = -g\sin\phi - a\cos\phi$$

$$l\ddot{\phi} = -\sqrt{g^2+a^2}\left(\frac{g}{\sqrt{g^2+a^2}}\sin\phi + \frac{a}{\sqrt{g^2+a^2}}\cos\phi\right)$$



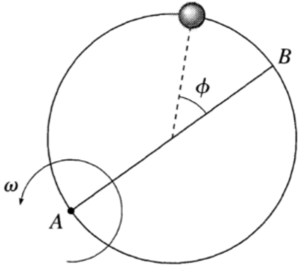
$$= -\sqrt{g^2+a^2}\left(\cos\beta\sin\phi + \sin\beta\cos\phi\right)$$

$$l\ddot{\phi} = -\sqrt{g^2+a^2}\sin(\phi+\beta)$$

let $\varepsilon = \phi - (-\beta) = \phi - \phi_0$

$$\ddot{\varepsilon} = \ddot{\phi} \qquad \ddot{\varepsilon} = -\frac{\sqrt{g^2+a^2}}{l}\sin\varepsilon$$

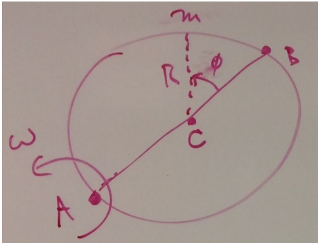
$$\rightarrow \omega_0 = \sqrt{\frac{\sqrt{g^2+a^2}}{l}}$$



The above-right figure is a bird's-eye view of a smooth horizontal wire hoop that is forced to rotate at a fixed angular frequency ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center with the diameter AB. Find the Lagrangian for this system using ϕ as your generalized coordinate. Use the Lagrange EOM to show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

Next slide shows a handy trick that is helpful when you're able to write $\vec{x}_m = \vec{x}_{\text{point}} + \vec{x}_{\text{relative}}$.

Next-next slide shows how to use Mathematica to eliminate the drudgery of calculating T.



$$\underset{\sim}{x}_c = R(\cos\omega t, \sin\omega t) \qquad \dot{\underset{\sim}{x}}_c = R\omega(-\sin\omega t, \cos\omega t)$$

$$\underset{\sim}{x}_m = \underset{\sim}{x}_c + \boxed{R\left(\cos(\phi+\omega t), \sin(\phi+\omega t)\right)}$$

$$\equiv \underset{\sim}{x}_c + \boxed{\underset{\sim}{x}_{relative}} \qquad \dot{\underset{\sim}{x}}_{rel} = R(\dot\phi+\omega)\left(-\sin(\phi+\omega t), \cos(\phi+\omega t)\right)$$

handy trick: $\left|\dot{\underset{\sim}{x}}_m\right|^2 = \left|\dot{\underset{\sim}{x}}_c + \dot{\underset{\sim}{x}}_{rel}\right|^2$

$$= \left|\dot{\underset{\sim}{x}}_c\right|^2 + \left|\dot{\underset{\sim}{x}}_{rel}\right|^2 + 2\,\dot{\underset{\sim}{x}}_c \cdot \dot{\underset{\sim}{x}}_{rel}$$

$$= (R\omega)^2 + \left(R(\dot\phi+\omega)\right)^2 + 2R^2\omega(\dot\phi+\omega)\left(\sin(\omega t)\sin(\phi+\omega t) + \cos(\omega t)\cos(\phi+\omega t)\right)$$

$$= R^2\omega^2 + R^2(\dot\phi+\omega)^2 + 2R^2\omega(\dot\phi+\omega)\cos\phi$$

```
]:= ClearAll["Global`*"];
    x[t_] := R Cos[ϕ[t] + ω t] + R Cos[ω t];
    y[t_] := R Sin[ϕ[t] + ω t] + R Sin[ω t];
    x'[t]^2 + y'[t]^2
```

]= $(R\,\omega\,\mathrm{Cos}[t\,\omega] + R\,\mathrm{Cos}[t\,\omega + \phi[t]]\,(\omega + \phi'[t]))^2 + (-R\,\omega\,\mathrm{Sin}[t\,\omega] - R\,\mathrm{Sin}[t\,\omega + \phi[t]]\,(\omega + \phi'[t]))^2$

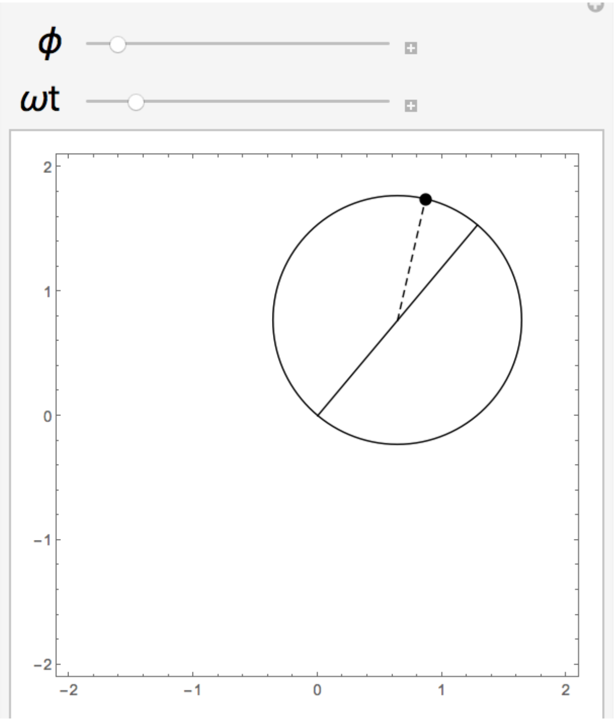
```
]:= FullSimplify[%]
```

1]= $R^2\left(2\,\omega^2\,(1 + \mathrm{Cos}[\phi[t]]) + 2\,\omega\,(1 + \mathrm{Cos}[\phi[t]])\,\phi'[t] + \phi'[t]^2\right)$

```
]:= Expand[%]
```

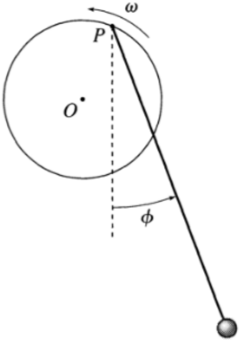
4]= $2\,R^2\,\omega^2 + 2\,R^2\,\omega^2\,\mathrm{Cos}[\phi[t]] + 2\,R^2\,\omega\,\phi'[t] + 2\,R^2\,\omega\,\mathrm{Cos}[\phi[t]]\,\phi'[t] + R^2\,\phi'[t]^2$

7]=



```
ClearAll["Global`*"];
ϕ = 0; ωt = 1;
Manipulate[
 r = 1; xc = r Cos[ωt]; yc = r Sin[ωt];
 c = {xc, yc}; a = {0, 0}; b = 2 c;
 xyrelative = r {Cos[ωt + ϕ], Sin[ωt + ϕ]};
 Graphics[{
   Circle[c, r],
   Line[{a, b}],
   Disk[c + xyrelative, 0.05 r],
   Dashed,
   Line[{c, c + xyrelative}]
  },
  PlotRange → {{-2.1, 2.1}, {-2.1, 2.1}},
  PlotRangeClipping → True, Frame → True],
 {ϕ, 0, 2 π}, {ωt, 0, 2 π},
 LabelStyle → Large]
```

The figure shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the EOM for the angle ϕ. [Hint: Be careful writing down T, the K.E. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case $\omega = 0$.



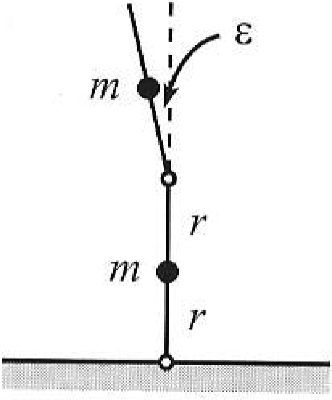
Next slide shows how to use Mathematica to eliminate the drudgery of calculating T.

```
In[6]:= ClearAll["Global`*"];
        x[t_] := R Cos[ω t] + l Sin[ϕ[t]];
        y[t_] := R Sin[ω t] - l Cos[ϕ[t]];
        x'[t]^2 + y'[t]^2
```

Out[9]= $(-R\,\omega\,\mathrm{Sin}[t\,\omega] + l\,\mathrm{Cos}[\phi[t]]\,\phi'[t])^2 + (R\,\omega\,\mathrm{Cos}[t\,\omega] + l\,\mathrm{Sin}[\phi[t]]\,\phi'[t])^2$

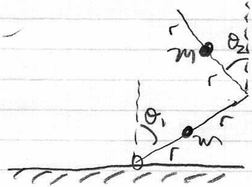
```
In[10]:= FullSimplify[%]
```

Out[10]= $R^2\,\omega^2 + l\,\phi'[t]\,(-2\,R\,\omega\,\mathrm{Sin}[t\,\omega - \phi[t]] + l\,\phi'[t])$



HW3 XC7 is the "three sticks" generalization of this problem. Let's try the "two sticks" version.

Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other. The bottom end of the lower stick is hinged on the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ε w.r.t. vertical. They are then released. At the instant after release, what are the angular accelerations of the two sticks? Work in the approximation where $\varepsilon \ll 1$.



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1 + y_2) = mgr(3\cos\theta_1 + \cos\theta_2)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2$$

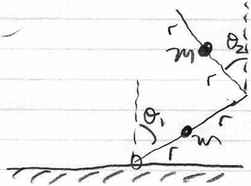
$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) =$$

$$= \frac{mr^2}{2}\Big[\cos^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_1\,\dot{\theta}_1^2 + 4\cos^2\theta_1\,\dot{\theta}_1^2 + \cos^2\theta_2\,\dot{\theta}_2^2 - 4\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2$$

$$+ 4\sin^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_2\,\dot{\theta}_2^2 + 4\sin\theta_1\sin\theta_2\,\dot{\theta}_1\dot{\theta}_2\Big]$$

$$T = \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1+\theta_2)\right] \simeq \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\left(1 - \frac{(\theta_1+\theta_2)^2}{2}\right)\right]$$



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1 + y_2) = mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\theta_2$$

$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$U = mgr\left(3\cos\theta_1 + \cos\theta_2\right) \simeq mgr\left(3 - \frac{3\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right)$$

$$\mathcal{L} = T - U = \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)\right) - mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$\simeq \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$L \simeq \frac{mr^2}{2}\left(5\dot\theta_1^2 + \dot\theta_2^2 - 4\dot\theta_1\dot\theta_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$\frac{\partial L}{\partial \theta_1} = 3mgr\theta_1 \qquad \frac{\partial L}{\partial \theta_2} = mgr\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(10\dot\theta_1 - 4\dot\theta_2\right)\right) = mr^2\left(5\ddot\theta_1 - 2\ddot\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(2\dot\theta_2 - 4\dot\theta_1\right)\right) = mr^2\left(\ddot\theta_2 - 2\ddot\theta_1\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \boxed{5\ddot\theta_1 - 2\ddot\theta_2 = \frac{3g\theta_1}{r}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \boxed{\ddot\theta_2 - 2\ddot\theta_1 = \frac{g\theta_2}{r}}$$

Now plug in, at $t = 0$, given conditions $\theta_1 = 0$, $\theta_2 = \varepsilon$, and find initial angular accelerations $\ddot\theta_1$ and $\ddot\theta_2$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \boxed{5\ddot\theta_1 - 2\ddot\theta_2 = \frac{3g\theta_1}{r}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \boxed{\ddot\theta_2 - 2\ddot\theta_1 = \frac{g\theta_2}{r}}$$

at $\boxed{t=0,}$ plug in $\theta_1 = 0, \quad \theta_2 = \varepsilon$

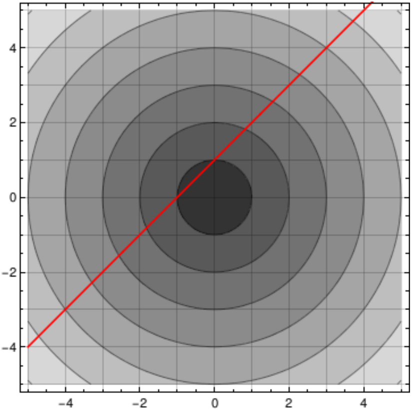
$$5\ddot\theta_1 = 2\ddot\theta_2 \qquad\qquad \ddot\theta_2 - 2\left(\frac{2}{5}\ddot\theta_2\right) = \frac{g\varepsilon}{r} = \frac{\ddot\theta_2}{5}$$

$$\rightarrow \ddot\theta_1 = \frac{2}{5}\ddot\theta_2 \qquad\qquad \rightarrow \boxed{\text{at } t=0, \quad \ddot\theta_2 = 5g\varepsilon/r \quad \ddot\theta_1 = 2g\varepsilon/r}$$

Math 114 problem: find the point (x, y) that minimizes

$$U(x, y) = mg\sqrt{x^2 + y^2}$$

subject to the constraint $y - x = 1$.



Let $f(x, y) = y - x - 1$. Then minimize the modified function

$$V(x, y) = U(x, y) + \lambda f(x, y)$$

w.r.t. variables x, y, and λ.

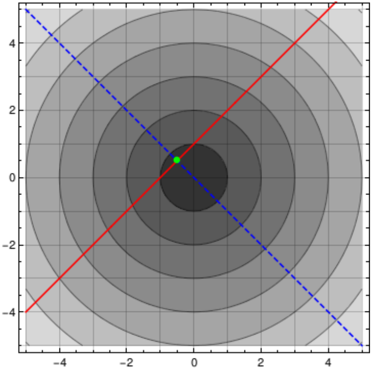
The added variable λ is called a **Lagrange multiplier**.

Let $f(x,y) = y - x - 1$. Then minimize the modified function

$$V(x,y) = U(x,y) + \lambda f(x,y)$$

w.r.t. variables x, y, and λ.

interpretation: notice $\nabla U \ \propto \ \nabla f$ — the two gradients are parallel, or antiparallel



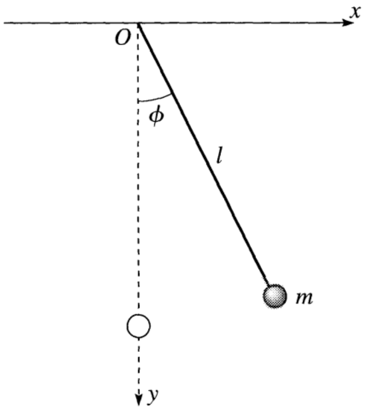
$$\left.\begin{aligned} 0 &= \frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{mgx}{\sqrt{x^2+y^2}} - \lambda \\ 0 &= \frac{\partial U}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{mgy}{\sqrt{x^2+y^2}} + \lambda \end{aligned}\right\} \frac{mg(x+y)}{\sqrt{x^2+y^2}} = 0$$

$$0 = \frac{\partial}{\partial \lambda}(U + \lambda f) = f(x,y) = 0 \Rightarrow y - x = 1$$

$$\Rightarrow (x,y) = \left(-\tfrac{1}{2}, +\tfrac{1}{2}\right) \qquad \lambda = \frac{mg}{2r}$$

(Taylor 7.51) Write down $\mathcal{L}$ for a pendulum in rectangular coordinates x and y, subject to

$$0 = f(x, y) = \sqrt{x^2 + y^2} - \ell$$



Write down the modified Lagrange equations.

Comparing with $\vec{F} = m\vec{a}$, show that λ is (minus) the tension in the rod.

Show that $\lambda \partial f / \partial x$ is the component of F_T in the x direction and that $\lambda \partial f / \partial y$ is the component of F_T in the y direction.

$$U = -mgy \qquad f(x,y) = \sqrt{x^2+y^2} - \ell$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy$$

$$\frac{\partial}{\partial x}(L + \lambda f) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}(L + \lambda f) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}\left(L+\lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

Newtonian approach: $m\ddot{x} = -F_T \sin\phi = -F_T \frac{x}{\ell}$

$$\Longrightarrow \boxed{\lambda = -F_T}$$

similarly, $mg - F_T \cos\phi = m\ddot{y}$

$$\lambda \frac{\partial f}{\partial x} = \left(-F_T\right)\left(\frac{x}{\ell}\right) = -F_T \sin\phi$$

$$\lambda \frac{\partial f}{\partial y} = \left(-F_T\right)\left(\frac{y}{\ell}\right) = -F_T \cos\phi$$

What if instead we had written $f(x, y) = x^2 + y^2 - \ell^2 = 0$? Try it!

You should find that λ itself no longer equals (in magnitude) the tension, but that it is still true that $\lambda\,\partial f/\partial x = F_{T,x}$ and that $\lambda\,\partial f/\partial y = F_{T,y}$.

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What if we had written $f(x,y) = x^2 + y^2 - \ell^2 = 0$?

$$\frac{\partial}{\partial x}(L + \lambda f) = 2\lambda x = m\ddot{x} \longrightarrow \lambda = \frac{m\ddot{x}}{2x} = -\frac{F_T}{2\ell}$$

$$\frac{\partial}{\partial y}(L + \lambda f) = mg + 2\lambda y = m\ddot{y}$$

$$\lambda \frac{\partial f}{\partial x} = \left(-\frac{F_T}{2\ell}\right)(2x) = -F_T \sin\phi \checkmark$$

$$\lambda \frac{\partial f}{\partial y} = \left(-\frac{F_T}{2\ell}\right)(2y) = -F_T \cos\phi \checkmark$$

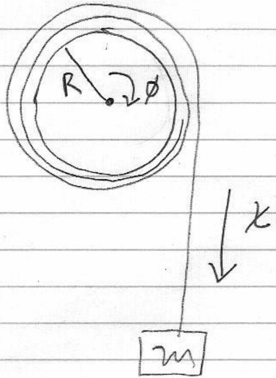
(Taylor 7.52) Lagrange multipliers also work with non-Cartesian coordinates. A mass m hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I) mounted on a frictionless horizontal axle. Let x be distance fallen by m, and let ϕ be angle wheel has turned.

Write modified Lagrange equations. Solve for $\ddot{x}$, for $\ddot{\phi}$, and for λ.

Use Newton's 2nd law to check $\ddot{x}$ and $\ddot{\phi}$.

Show that $\lambda \, \partial f / \partial x = F_{T,x}$.

What is your interpretation of the quantity $\lambda \, \partial f / \partial \phi$?



$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -mgx$$

$$0 = f(x, \phi) = R\phi - x$$

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + mgx$$

$$\frac{d}{dx}(L + \lambda f) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right)$$

$$\frac{d}{d\phi}(L + \lambda f) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right)$$

$$\frac{d}{d\lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I(\ddot{x}/R) \rightarrow \lambda = \ddot{x} I/R^2$$

$$\frac{\partial}{\partial x}(L+\lambda f) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial}{\partial \phi}(L+\lambda f) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$\frac{\partial}{\partial \lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I(\ddot{x}/R) \rightarrow \lambda = \ddot{x} I/R^2$$

$$mg - \frac{\ddot{x} I}{R^2} = m\ddot{x} \longrightarrow \ddot{x}\left(1+\frac{I}{mR^2}\right) = g$$

$$\ddot{x} = \frac{g}{1+(I/mR^2)}$$

$$\lambda = \ddot{x} I/R^2 = \frac{gI/R^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} = mg\left(\frac{I}{I+mR^2}\right)$$

$$\lambda \frac{\partial f}{\partial x} = -mg\left(\frac{I}{I+mR^2}\right) \qquad \lambda \frac{\partial f}{\partial \phi} = +mgR\left(\frac{I}{I+mR^2}\right)$$

Newton: $ma = mg - F_T \rightarrow F_T = m(g-\ddot{x}) = mg - \dfrac{mg}{1+I/mR^2}$

$$F_T = mg\,\frac{1+(I/mR^2)-1}{1+I/mR^2} = mg\,\frac{I/mR^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} \checkmark$$

Physics 351 — Wednesday, February 7, 2018

- HW3 due Friday. You finished reading ch7 last weekend. You'll read ch8 (Kepler problem) this weekend.
- HW help: Bill is in DRL 3N6 Wednesdays 4pm–7pm. Grace is in DRL 2C2 Thursdays 5:30pm-8:30pm.