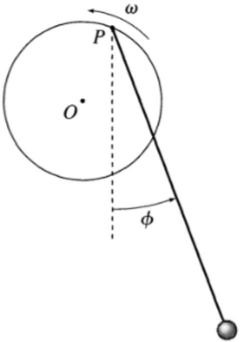
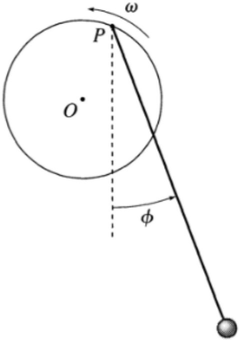
Physics 351 — Friday, February 9, 2018

- Turn in HW3. Pick up HW4 handout. Read ch8 (Kepler problem) this weekend, and answer online questions. When reading ch8, focus on the big picture, not the many details.
- If you catch me starting to write on unreadable parts of the board, please yell out to correct me immediately!

The figure shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the EOM for the angle ϕ. [Hint: Be careful writing down T, the K.E. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case $\omega = 0$.



The figure shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the EOM for the angle ϕ. [Hint: Be careful writing down T, the K.E. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case $\omega = 0$.



Next slide shows how to use Mathematica to eliminate the drudgery of calculating T.

In[6]:=
```
ClearAll["Global`*"];
x[t_] := R Cos[ω t] + l Sin[ϕ[t]];
y[t_] := R Sin[ω t] - l Cos[ϕ[t]];
x'[t]^2 + y'[t]^2
```

Out[9]=
```
(-R ω Sin[t ω] + l Cos[ϕ[t]] ϕ′[t])^2 +
  (R ω Cos[t ω] + l Sin[ϕ[t]] ϕ′[t])^2
```

In[10]:=
```
FullSimplify[%]
```

Out[10]=
```
R^2 ω^2 + l ϕ′[t] (-2 R ω Sin[t ω - ϕ[t]] + l ϕ′[t])
```

Here's a chance for us to practice making approximations:

A spring with spring constant k and relaxed length zero lies along a spoke of a massless wheel of radius R. One end of the spring is attached to the center, and the other end is attached to a mass m that is free to slide along the spoke. When the system is in its equilibrium position with the spring hanging vertically, how far (in terms of R) should the mass hang down (you are free to adjust k) so that for small oscillations, the frequency of the spring oscillations equals the frequency of the rocking motion of the wheel? Assume that the wheel rolls without slipping. (Once you let $r = r_0 + \epsilon$, you should find that $r_0 = mg/k = R/2$.)

We'll try using Mathematica to reduce the drudgery. Theorists who make higher-order calculations to be compared with precise experimental measurements don't just work with pencil and paper.

You don't routinely do long division by hand: you learn how to do it, then practice until you can do it reliably, then switch to letting a machine do it for you!

```
ClearAll["Global`*"];
xcenter[t_] := R ϕ[t];
ym[t_] := - r[t] Cos[ϕ[t]];
xm[t_] := xcenter[t] - r[t] Sin[ϕ[t]];
U = m g ym[t] + (1 / 2) k r[t] ^2;
T = (1 / 2) m (xm'[t] ^2 + ym'[t] ^2);
L = T - U;
FullSimplify[L]
```

$$g\, m \operatorname{Cos}[\phi[t]]\, r[t] - \frac{1}{2} k\, r[t]^2 + \frac{1}{2} m \left(r'[t]^2 - 2 R \operatorname{Sin}[\phi[t]]\, r'[t]\, \phi'[t] + \left(R^2 - 2 R \operatorname{Cos}[\phi[t]]\, r[t] + r[t]^2\right) \phi'[t]^2\right)$$

A spring with spring constant k and relaxed length zero lies along a spoke of a massless wheel of radius R. One end of the spring is attached to the center, and the other end is attached to a mass m that is free to slide along the spoke. When the system is in its equilibrium position with the spring hanging vertically, how far (in terms of R) should the mass hang down (you are free to adjust k) so that for small oscillations, the frequency of the spring oscillations equals the frequency of the rocking motion of the wheel? Assume that the wheel rolls without slipping. (Once you let $r = r_0 + \epsilon$, you should find that $r_0 = mg/k = R/2$.)

One easy mistake is forgetting to write $r[t]$ instead of just r.

$$\mathcal{L} = mgr\cos\phi - \frac{1}{2}kr^2 + \frac{1}{2}m(\dot{r}^2 - 2R\dot{r}\dot{\phi}\sin\phi) + \frac{1}{2}m\dot{\phi}^2(R^2 + r^2 - 2Rr\cos\phi)$$

```
In[]:= eom1 = D[L, ϕ[t]] == D[ D[L, ϕ'[t]], t];
FullSimplify[eom1]
```

```
m (g r[t] Sin[ϕ[t]] - 2 R Cos[ϕ[t]] r′[t] ϕ′[t] + 2 r[t] r′[t] ϕ′[t] +
    R r[t] Sin[ϕ[t]] ϕ′[t]^2 - R Sin[ϕ[t]] r″[t] +
    (R^2 - 2 R Cos[ϕ[t]] r[t] + r[t]^2) ϕ″[t]) == 0
```

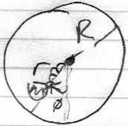
```
In[]:= eom2 = D[L, r[t]] == D[ D[L, r'[t]], t];
FullSimplify[eom2]
```

```
Out[]= k r[t] + m r″[t] == m (g Cos[ϕ[t]] + r[t] ϕ′[t]^2 + R Sin[ϕ[t]] ϕ″[t])
```

$$gr\sin\phi - 2R\dot{r}\dot{\phi}\cos\phi + 2r\dot{r}\dot{\phi} + Rr\dot{\phi}^2\sin\phi - R\ddot{r}\sin\phi + \\ (R^2 - 2Rr\cos\phi + r^2)\ddot{\phi} = 0$$

$$\ddot{r} + \frac{k}{m}r = g\cos\phi + r\dot{\phi}^2 + r\ddot{\phi}\sin\phi$$

$x_c = R\phi$



$x_m = x_c - r\sin\phi$

$y_m = -r\cos\phi$

$U = -mgr\cos\phi + \frac{1}{2}kr^2 \qquad T = \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right)$

$$\mathcal{L} = mgr\cos\phi - \frac{1}{2}kr^2 + \frac{1}{2}m\left(\dot{r}^2 - 2R\dot{r}\dot{\phi}\sin\phi\right) + \frac{1}{2}m\dot{\phi}^2\left(R^2 + r^2 - 2Rr\cos\phi\right)$$

$$\ddot{r} + \frac{k}{m}r = g\cos\phi + r\dot{\phi}^2 + r\ddot{\phi}\sin\phi \simeq g$$

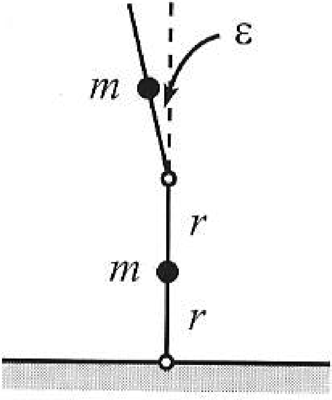
let $r = r_0 + \varepsilon$ with $r_0 = \frac{mg}{k}$ $\Rightarrow$ $\boxed{\ddot{\varepsilon} = -\frac{k}{m}\varepsilon}$

$$gr\sin\phi - 2R\dot{r}\dot{\phi}\cos\phi + 2r\dot{r}\dot{\phi} + Rr\dot{\phi}^2\sin\phi - R\ddot{r}\sin\phi + \left(R^2 - 2Rr\cos\phi + r^2\right)\ddot{\phi} = 0$$

$$gr_0\phi + \left(R^2 - 2Rr_0 + r_0^2\right)\ddot{\phi} = 0 = gr_0\phi + (R - r_0)^2\ddot{\phi}$$

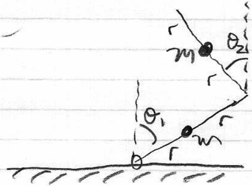
$\boxed{\ddot{\phi} = -\frac{gr_0}{(R-r_0)^2}\phi}$ want $\frac{g}{r_0} = \frac{gr_0}{(R-r_0)^2}$

$r_0^2 = (R - r_0)^2 = R^2 - 2r_0R + r_0^2 \Rightarrow \boxed{R = 2r_0}$



HW3 XC7 is the "three sticks" generalization of this problem. Let's try the "two sticks" version.

Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other. The bottom end of the lower stick is hinged on the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ε w.r.t. vertical. They are then released. At the instant after release, what are the angular accelerations of the two sticks? Work in the approximation where $\varepsilon \ll 1$.



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1+y_2) = mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

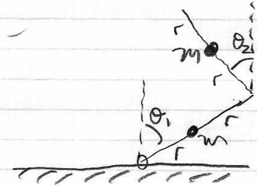
$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2$$

$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) =$$

$$= \frac{mr^2}{2}\Big[\cos^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_1\,\dot{\theta}_1^2 + 4\cos^2\theta_1\,\dot{\theta}_1^2 + \cos^2\theta_2\,\dot{\theta}_2^2 - 4\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2$$
$$+ 4\sin^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_2\,\dot{\theta}_2^2 + 4\sin\theta_1\sin\theta_2\,\dot{\theta}_1\dot{\theta}_2\Big]$$

$$T = \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1+\theta_2)\right] \simeq \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\left(1 - \frac{(\theta_1+\theta_2)^2}{2}\right)\right]$$



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1+y_2) = mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2$$

$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$U = mgr\left(3\cos\theta_1 + \cos\theta_2\right) \simeq mgr\left(3 - \frac{3\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right)$$

$$\mathcal{L} = T - U = \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1+\theta_2)\right) - mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$\simeq \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$L \simeq \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$\frac{\partial L}{\partial \theta_1} = 3mgr\theta_1 \qquad \frac{\partial L}{\partial \theta_2} = mgr\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(10\dot{\theta}_1 - 4\dot{\theta}_2\right)\right) = mr^2\left(5\ddot{\theta}_1 - 2\ddot{\theta}_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(2\dot{\theta}_2 - 4\dot{\theta}_1\right)\right) = mr^2\left(\ddot{\theta}_2 - 2\ddot{\theta}_1\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \boxed{5\ddot{\theta}_1 - 2\ddot{\theta}_2 = \frac{3g\theta_1}{r}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \boxed{\ddot{\theta}_2 - 2\ddot{\theta}_1 = \frac{g\theta_2}{r}}$$

Now plug in, at $t = 0$, given conditions $\theta_1 = 0$, $\theta_2 = \varepsilon$, and find initial angular accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \quad 5\ddot\theta_1 - 2\ddot\theta_2 = \frac{3g\theta_1}{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \quad \ddot\theta_2 - 2\ddot\theta_1 = \frac{g\theta_2}{r}$$

at $\boxed{t=0,}$ plug in $\theta_1 = 0, \ \theta_2 = \varepsilon$

$$5\ddot\theta_1 = 2\ddot\theta_2$$

$$\rightarrow \ddot\theta_1 = \frac{2}{5}\ddot\theta_2$$

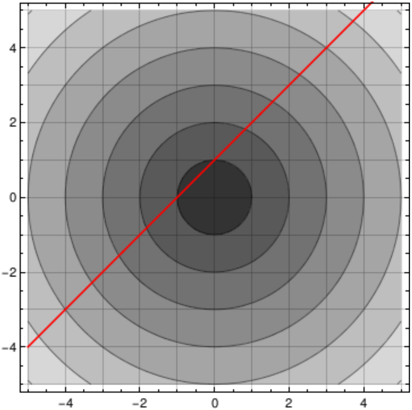
$$\ddot\theta_2 - 2\left(\frac{2}{5}\ddot\theta_2\right) = \frac{g\varepsilon}{r} = \frac{\ddot\theta_2}{5}$$

$$\rightarrow \boxed{\text{at } t=0, \quad \begin{aligned} \ddot\theta_2 &= 5g\varepsilon/r \\ \ddot\theta_1 &= 2g\varepsilon/r \end{aligned}}$$

Math 114 problem: find the point (x, y) that minimizes

$$U(x, y) = mg\sqrt{x^2 + y^2}$$

subject to the constraint $y - x = 1$.



Let $f(x, y) = y - x - 1$. Then minimize the modified function

$$V(x, y) = U(x, y) + \lambda f(x, y)$$

w.r.t. variables x, y, and λ.

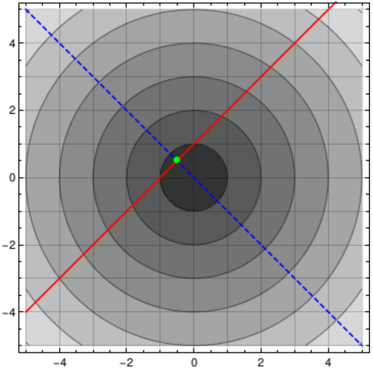
The added variable λ is called a **Lagrange multiplier**.

Let $f(x,y) = y - x - 1$. Then minimize the modified function

$$V(x,y) = U(x,y) + \lambda f(x,y)$$

w.r.t. variables x, y, and λ.

interpretation: notice $\nabla U \propto \nabla f$ — the two gradients are parallel, or antiparallel



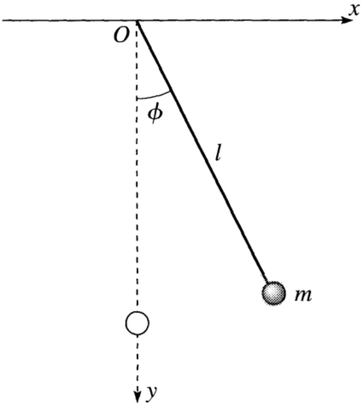
$$\left.\begin{aligned} 0 &= \frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{mgx}{\sqrt{x^2+y^2}} - \lambda \\ 0 &= \frac{\partial U}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{mgy}{\sqrt{x^2+y^2}} + \lambda \end{aligned}\right\} \frac{mg(x+y)}{\sqrt{x^2+y^2}} = 0$$

$$0 = \frac{\partial}{\partial \lambda}\left(U + \lambda f\right) = f(x,y) = 0 \Rightarrow y - x = 1$$

$$\Rightarrow (x,y) = \left(-\tfrac{1}{2}, +\tfrac{1}{2}\right) \qquad \lambda = \frac{mg}{2r}$$

(Taylor 7.51) Write down $\mathcal{L}$ for a pendulum in rectangular coordinates x and y, subject to

$$0 = f(x, y) = \sqrt{x^2 + y^2} - \ell$$



Write down the modified Lagrange equations.

Comparing with $\vec{F} = m\vec{a}$, show that λ is (minus) the tension in the rod.

Show that $\lambda \partial f/\partial x$ is the component of F_T in the x direction and that $\lambda \partial f/\partial y$ is the component of F_T in the y direction.

$$U = -mgy \qquad f(x,y) = \sqrt{x^2+y^2} - \ell$$

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + mgy$$

$$\frac{\partial}{\partial x}\left(L + \lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}\left(L + \lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}\left(L + \lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

Newtonian approach: $m\ddot{x} = -F_T \sin\phi = -F_T \frac{x}{\ell}$

$$\Longrightarrow \boxed{\lambda = -F_T}$$

similarly, $mg - F_T\cos\phi = m\ddot{y}$

$$\lambda \frac{\partial f}{\partial x} = \left(-F_T\right)\left(\frac{x}{\ell}\right) = -F_T \sin\phi$$

$$\lambda \frac{\partial f}{\partial y} = \left(-F_T\right)\left(\frac{y}{\ell}\right) = -F_T \cos\phi$$

What if instead we had written $f(x, y) = x^2 + y^2 - \ell^2 = 0$? Try it!

You should find that λ itself no longer equals (in magnitude) the tension, but that it is still true that $\lambda\, \partial f / \partial x = F_{T,x}$ and that $\lambda\, \partial f / \partial y = F_{T,y}$.

What if instead we had written $f(x,y) = x^2 + y^2 - \ell^2 = 0$? Try it!

You should find that λ itself no longer equals (in magnitude) the tension, but that it is still true that $\lambda\,\partial f/\partial x = F_{T,x}$ and that $\lambda\,\partial f/\partial y = F_{T,y}$.

What if we had written $f(x,y) = x^2+y^2-\ell^2 = 0$?

$$\frac{\partial}{\partial x}(L+\lambda f) = 2\lambda x = m\ddot{x} \longrightarrow \lambda = \frac{m\ddot{x}}{2x} = -\frac{F_T}{2\ell}$$

$$\frac{\partial}{\partial y}(L+\lambda f) = mg + 2\lambda y = m\ddot{y}$$

$$\lambda\frac{\partial f}{\partial x} = \left(-\frac{F_T}{2\ell}\right)(2x) = -F_T\sin\phi \quad \checkmark$$

$$\lambda\frac{\partial f}{\partial y} = \left(-\frac{F_T}{2\ell}\right)(2y) = -F_T\cos\phi \quad \checkmark$$

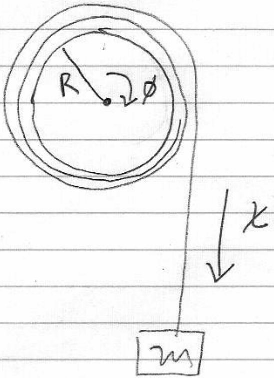
(Taylor 7.52) Lagrange multipliers also work with non-Cartesian coordinates. A mass m hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I) mounted on a frictionless horizontal axle. Let x be distance fallen by m, and let ϕ be angle wheel has turned.

Write modified Lagrange equations. Solve for $\ddot{x}$, for $\ddot{\phi}$, and for λ.

Use Newton's 2nd law to check $\ddot{x}$ and $\ddot{\phi}$.

Show that $\lambda\,\partial f/\partial x = F_{T,x}$.

What is your interpretation of the quantity $\lambda\,\partial f/\partial\phi$?



$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -mgx$$

$$0 = f(x,\phi) = R\phi - x$$

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + mgx$$

$$\frac{d}{dx}\left(L + \lambda f\right) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right)$$

$$\frac{d}{d\phi}\left(L + \lambda f\right) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right)$$

$$\frac{d}{d\lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I\left(\ddot{x}/R\right) \rightarrow \lambda = \ddot{x}\, I/R^2$$

$$\frac{\partial}{\partial x}(L+\lambda f) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial}{\partial \phi}(L+\lambda f) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$\frac{\partial}{\partial \lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I(\ddot{x}/R) \rightarrow \lambda = \ddot{x}I/R^2$$

$$mg - \frac{\ddot{x}I}{R^2} = m\ddot{x} \longrightarrow \ddot{x}\left(1+\frac{I}{mR^2}\right) = g$$

$$\ddot{x} = \frac{g}{1+(I/mR^2)}$$

$$\lambda = \ddot{x}I/R^2 = \frac{gI/R^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} = mg\left(\frac{I}{I+mR^2}\right)$$

$$\lambda\frac{\partial f}{\partial x} = -mg\left(\frac{I}{I+mR^2}\right) \qquad \lambda\frac{\partial f}{\partial \phi} = +mgR\left(\frac{I}{I+mR^2}\right)$$

Newton: $ma = mg - F_T \rightarrow F_T = m(g-\ddot{x}) = mg - \dfrac{mg}{1+I/mR^2}$

$$F_T = mg\,\frac{1+(I/mR^2)-1}{1+I/mR^2} = mg\,\frac{I/mR^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} \checkmark$$

Physics 351 — Friday, February 9, 2018

- Turn in HW3. Pick up HW4 handout.
- Read Chapter 8 (two-body central-force problems) for Monday. Chapter 8 derives a huge number of detailed results about Kepler orbits, only a few of which are worth remembering. Focus instead on the big ideas, the neat application of Lagrangian mechanics, the conserved quantities, the impressive step-by-step reduction of a 6-coordinate problem to a 1-coordinate problem.
  - Here's a comment from a spring 2015 student:
  - "I found the presence of many different equations for the many different features of orbits to be slightly overwhelming. This section feels a little like the air-resistance section, where there were many formulas describing many behaviors."
  - For air resistance, detailed results were not so important, but qualitative results, separation of variables, etc. were useful.