Physics 351 — Monday, February 12, 2018

- You read ch8 (Kepler problem) this past weekend. We'll discuss it Wednesday.
- If you catch me starting to write on unreadable parts of the board, please yell out to correct me immediately!

Math 114 problem: find the point (x, y) that minimizes

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subject to the constraint $y - x = 1$.

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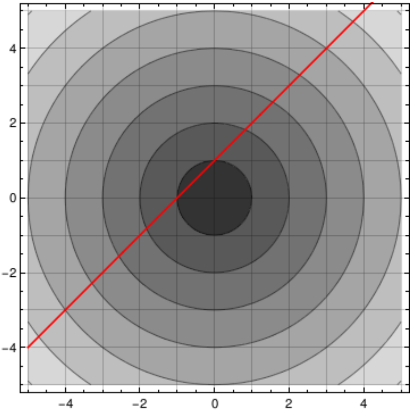

Let $f(x, y) = y - x - 1$. Then minimize the modified function

$$V(x, y) = U(x, y) + \lambda f(x, y)$$

w.r.t. variables x, y, and λ.

The added variable λ is called a **Lagrange multiplier**.

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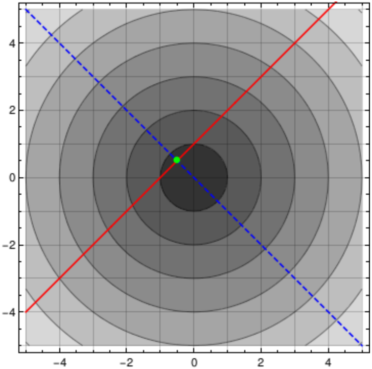
interpretation: notice $\nabla U \ \propto \ \nabla f$ — the two gradients are parallel, or antiparallel



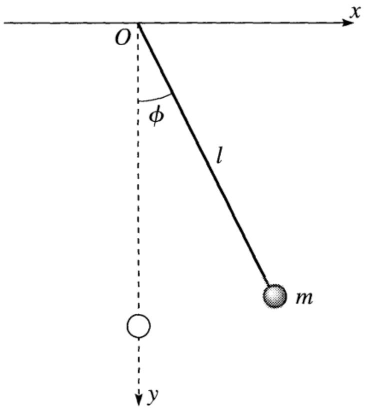
$$\left.\begin{aligned} 0 &= \frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{mgx}{\sqrt{x^2+y^2}} - \lambda \\ 0 &= \frac{\partial U}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{mgy}{\sqrt{x^2+y^2}} + \lambda \end{aligned}\right\} \frac{mg(x+y)}{\sqrt{x^2+y^2}} = 0$$

$$0 = \frac{\partial}{\partial \lambda}(U + \lambda f) = f(x,y) = 0 \Rightarrow y - x = 1$$

$$\Rightarrow (x,y) = \left(-\tfrac{1}{2}, +\tfrac{1}{2}\right) \qquad \lambda = \frac{mg}{2r}$$

(Taylor 7.51) Write down $\mathcal{L}$ for a pendulum in rectangular coordinates x and y, subject to

$$0 = f(x, y) = \sqrt{x^2 + y^2} - \ell$$



Write down the modified Lagrange equations.

Comparing with $\vec{F} = m\vec{a}$, show that λ is (minus) the tension in the rod.

Show that $\lambda \partial f/\partial x$ is the component of F_T in the x direction and that $\lambda \partial f/\partial y$ is the component of F_T in the y direction.

$$U = -mgy \qquad f(x,y) = \sqrt{x^2+y^2} - \ell$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy$$

$$\frac{\partial}{\partial x}\left(L + \lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}\left(L + \lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

$$\frac{\lambda x}{\sqrt{x^2+y^2}} = m\ddot{x} \longrightarrow \frac{\lambda x}{\ell} = m\ddot{x}$$

$$\frac{\partial}{\partial y}\left(L+\lambda f\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$mg + \frac{\lambda y}{\sqrt{x^2+y^2}} = m\ddot{y} \longrightarrow mg + \frac{\lambda y}{\ell} = m\ddot{y}$$

Newtonian approach: $m\ddot{x} = -F_T \sin\phi = -F_T \frac{x}{\ell}$

$$\Longrightarrow \boxed{\lambda = -F_T}$$

similarly, $mg - F_T\cos\phi = m\ddot{y}$

$$\lambda \frac{\partial f}{\partial x} = (-F_T)\left(\frac{x}{\ell}\right) = -F_T \sin\phi$$

$$\lambda \frac{\partial f}{\partial y} = (-F_T)\left(\frac{y}{\ell}\right) = -F_T \cos\phi$$

What if instead we had written $f(x, y) = x^2 + y^2 - \ell^2 = 0$? Try it!

You should find that λ itself no longer equals (in magnitude) the tension, but that it is still true that $\lambda \, \partial f / \partial x = F_{T,x}$ and that $\lambda \, \partial f / \partial y = F_{T,y}$.

What if instead we had written $f(x, y) = x^2 + y^2 - \ell^2 = 0$? Try it!

You should find that λ itself no longer equals (in magnitude) the tension, but that it is still true that $\lambda\,\partial f/\partial x = F_{T,x}$ and that $\lambda\,\partial f/\partial y = F_{T,y}$.

What if we had written $f(x,y) = x^2 + y^2 - \ell^2 = 0$?

$$\frac{\partial}{\partial x}(L + \lambda f) = 2\lambda x = m\ddot{x} \longrightarrow \lambda = \frac{m\ddot{x}}{2x} = -\frac{F_T}{2\ell}$$

$$\frac{\partial}{\partial y}(L + \lambda f) = mg + 2\lambda y = m\ddot{y}$$

$$\lambda \frac{\partial f}{\partial x} = \left(-\frac{F_T}{2\ell}\right)(2x) = -F_T \sin\phi \quad \checkmark$$

$$\lambda \frac{\partial f}{\partial y} = \left(-\frac{F_T}{2\ell}\right)(2y) = -F_T \cos\phi \quad \checkmark$$

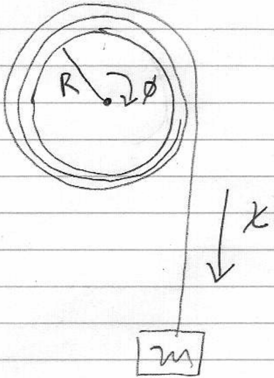
(Taylor 7.52) Lagrange multipliers also work with non-Cartesian coordinates. A mass m hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I) mounted on a frictionless horizontal axle. Let x be distance fallen by m, and let ϕ be angle wheel has turned.

Write modified Lagrange equations. Solve for $\ddot{x}$, for $\ddot{\phi}$, and for λ.

Use Newton's 2nd law to check $\ddot{x}$ and $\ddot{\phi}$.

Show that $\lambda\,\partial f/\partial x = F_{T,x}$.

What is your interpretation of the quantity $\lambda\,\partial f/\partial \phi$?



$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -mgx$$

$$0 = f(x, \phi) = R\phi - x$$

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + mgx$$

$$\frac{d}{dx}\left(L + \lambda f\right) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right)$$

$$\frac{d}{d\phi}\left(L + \lambda f\right) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right)$$

$$\frac{d}{d\lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I\left(\ddot{x}/R\right) \rightarrow \lambda = \ddot{x} I / R^2$$

$$\frac{\partial}{\partial x}(L+\lambda f) = +mg-\lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial}{\partial \phi}(L+\lambda f) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$\frac{d}{d\lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I(\ddot{x}/R) \rightarrow \lambda = \ddot{x}I/R^2$$

$$mg - \frac{\ddot{x}I}{R^2} = m\ddot{x} \longrightarrow \ddot{x}\left(1+\frac{I}{mR^2}\right) = g$$

$$\ddot{x} = \frac{g}{1+(I/mR^2)}$$

$$\lambda = \ddot{x}I/R^2 = \frac{gI/R^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} = mg\left(\frac{I}{I+mR^2}\right)$$

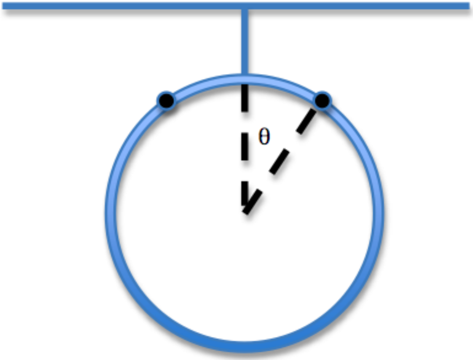
$$\lambda\frac{\partial f}{\partial x} = -mg\left(\frac{I}{I+mR^2}\right) \qquad \lambda\frac{\partial f}{\partial \phi} = +mgR\left(\frac{I}{I+mR^2}\right)$$

Newton: $ma = mg - F_T \rightarrow F_T = m(g-\ddot{x}) = mg - \dfrac{mg}{1+I/mR^2}$

$$F_T = mg\,\frac{1+(I/mR^2)-1}{1+I/mR^2} = mg\,\frac{I/mR^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} \quad \checkmark$$

A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction. The beads are released simultaneously from rest at the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > \frac{3}{2}M$, and find the angle θ at which this occurs. (If $M = 0$ then $\cos\theta = \frac{2}{3}$.)

Write $\mathcal{L}(\theta, \dot{\theta}, Y, \dot{Y})$ and include Lagrange multiplier term λY to enforce the $Y = 0$ constraint. "The ring starts to rise" implies $\lambda = 0$, i.e. string tension is zero.



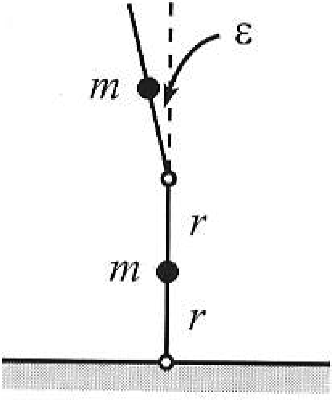
```
ClearAll["Global`*"]; y[t_] := Y[t] + R Cos[θ[t]];
x[t_] := R Sin[θ[t]];
T = m (x'[t]^2 + y'[t]^2) + (1 / 2) M Y'[t]^2;
U = 2 m g y[t] + M g Y[t];
lag = T - U
```

```
FullSimplify[lag]
```

```
lagWithConstraint = lag + λ Y[t];
FullSimplify[
 D[D[lagWithConstraint, Y'[t]], t] == D[lagWithConstraint, Y[t]]
]
```

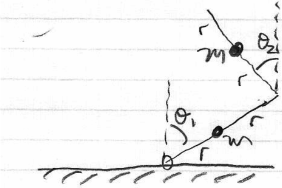
```
FullSimplify[
 D[D[lagWithConstraint, θ'[t]], t] == D[lagWithConstraint, θ[t]]
]
```

Now you can impose the constraint that $Y \equiv 0$, exploit the helpful fact that mechanical energy of each bead is constant (as long as $Y \equiv 0$), and solve for the condition that the force of constraint equals zero when the ring just barely starts to rise. You'll get a quadratic equation for $\cos\theta$ in terms of the ratio $3M/2m$.



HW3 XC7 is the "three sticks" generalization of this problem. Let's try the "two sticks" version.

Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other. The bottom end of the lower stick is hinged on the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ε w.r.t. vertical. They are then released. At the instant after release, what are the angular accelerations of the two sticks? Work in the approximation where $\varepsilon \ll 1$.



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1 + y_2) = mgr\left(3\cos\theta_1 + \cos\theta_2\right)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

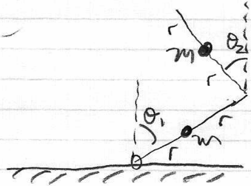
$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2$$

$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) =$$

$$= \frac{mr^2}{2}\Big[\cos^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_1\,\dot{\theta}_1^2 + 4\cos^2\theta_1\,\dot{\theta}_1^2 + \cos^2\theta_2\,\dot{\theta}_2^2 - 4\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2$$
$$+ 4\sin^2\theta_1\,\dot{\theta}_1^2 + \sin^2\theta_2\,\dot{\theta}_2^2 + 4\sin\theta_1\sin\theta_2\,\dot{\theta}_1\dot{\theta}_2\Big]$$

$$T = \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1+\theta_2)\right] \simeq \frac{mr^2}{2}\left[5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\left(1 - \frac{(\theta_1+\theta_2)^2}{2}\right)\right]$$



$$y_1 = r\cos\theta_1$$

$$y_2 = 2r\cos\theta_1 + r\cos\theta_2$$

$$U = mg(y_1 + y_2) = mgr(3\cos\theta_1 + \cos\theta_2)$$

$$x_1 = r\sin\theta_1 \rightarrow \dot{x}_1 = r\cos\theta_1\,\dot{\theta}_1$$

$$x_2 = 2r\sin\theta_1 - r\sin\theta_2 \rightarrow \dot{x}_2 = 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2$$

$$\dot{y}_1 = -r\sin\theta_1\,\dot{\theta}_1 \qquad \dot{y}_2 = -2r\sin\theta_1\,\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2$$

$$U = mgr(3\cos\theta_1 + \cos\theta_2) \simeq mgr\left(3 - \frac{3\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right)$$

$$\mathcal{L} = T - U = \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)\right) - mgr(3\cos\theta_1 + \cos\theta_2)$$

$$\simeq \frac{mr^2}{2}\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 - 4\dot{\theta}_1\dot{\theta}_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$L \simeq \frac{mr^2}{2}\left(5\dot\theta_1^2 + \dot\theta_2^2 - 4\dot\theta_1\dot\theta_2\right) + mgr\left(\frac{3\theta_1^2}{2} + \frac{\theta_2^2}{2} - 4\right)$$

$$\frac{\partial L}{\partial \theta_1} = 3mgr\theta_1 \qquad \frac{\partial L}{\partial \theta_2} = mgr\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(10\dot\theta_1 - 4\dot\theta_2\right)\right) = mr^2\left(5\ddot\theta_1 - 2\ddot\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = \frac{d}{dt}\left(\frac{mr^2}{2}\left(2\dot\theta_2 - 4\dot\theta_1\right)\right) = mr^2\left(\ddot\theta_2 - 2\ddot\theta_1\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \boxed{5\ddot\theta_1 - 2\ddot\theta_2 = \frac{3g\theta_1}{r}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \boxed{\ddot\theta_2 - 2\ddot\theta_1 = \frac{g\theta_2}{r}}$$

Now plug in, at $t = 0$, given conditions $\theta_1 = 0$, $\theta_2 = \varepsilon$, and find initial angular accelerations $\ddot\theta_1$ and $\ddot\theta_2$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \quad 5\ddot{\theta}_1 - 2\ddot{\theta}_2 = \frac{3g\theta_1}{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial L}{\partial \theta_2} \Rightarrow \quad \ddot{\theta}_2 - 2\ddot{\theta}_1 = \frac{g\theta_2}{r}$$

at $\boxed{t=0,}$ plug in $\theta_1 = 0, \quad \theta_2 = \varepsilon$

$$5\ddot{\theta}_1 = 2\ddot{\theta}_2 \qquad \ddot{\theta}_2 - 2\left(\frac{2}{5}\ddot{\theta}_2\right) = \frac{g\varepsilon}{r} = \frac{\ddot{\theta}_2}{5}$$

$$\rightarrow \ddot{\theta}_1 = \frac{2}{5}\ddot{\theta}_2 \qquad \rightarrow \boxed{\text{at } t=0, \quad \ddot{\theta}_2 = 5g\varepsilon/r, \quad \ddot{\theta}_1 = 2g\varepsilon/r}$$

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