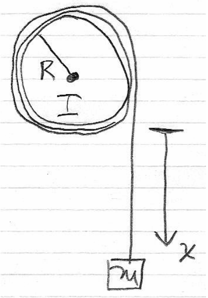
Physics 351 — Wednesday, February 14, 2018

- HW4 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm.
- Respond at `pollev.com/phys351` or text `PHYS351` to 37607 once to join, then A, B, or C.

A mass m hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I). The wheel is mounted on a frictionless horizontal axle. The string is wound tightly and does not slip, so the wheel must turn as the string lengthens. Let x be the distance fallen by m. Using $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I(\dot{x}/R)^2$ and $U = -mgx$, write the Lagrangian and find the EOM for $\ddot{x}$.



(A) $\ddot{x} = g$ (B) $\ddot{x} = \frac{g}{1 + (mR^2)/I}$ (C) $\ddot{x} = \frac{g}{1 + I/(mR^2)}$

Now suppose we wanted, in the same Lagrangian problem you just solved, to find the tension in the string? Annoyingly, the "Lagrange multiplier" formalism now requires us to solve for three unknowns ($\ddot{x}$, $\ddot{\phi}$, and λ) instead of just one. Let's try it.

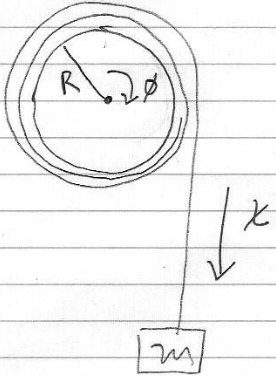
(Taylor 7.52) Lagrange multipliers also work with non-Cartesian coordinates. A mass m hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I) mounted on a frictionless horizontal axle. Let x be distance fallen by m, and let ϕ be angle wheel has turned.

Write modified Lagrange equations. Solve for $\ddot{x}$, for $\ddot{\phi}$, and for λ.

Use Newton's 2nd law to check $\ddot{x}$ and $\ddot{\phi}$. (You already did, before class.)

Show that $\lambda \, \partial f / \partial x = F_{T,x}$.

What is your interpretation of the quantity $\lambda \, \partial f / \partial \phi$?



$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -mgx$$

$$0 = f(x, \phi) = R\phi - x$$

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + mgx$$

$$\frac{d}{dx}\left(L + \lambda f\right) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right)$$

$$\frac{d}{d\phi}\left(L + \lambda f\right) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right)$$

$$\frac{d}{d\lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I\left(\ddot{x}/R\right) \rightarrow \lambda = \ddot{x}\, I/R^2$$

$$\frac{\partial}{\partial x}(L+\lambda f) = +mg - \lambda = m\ddot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial}{\partial \phi}(L+\lambda f) = \lambda R = I\ddot{\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$\frac{\partial}{\partial \lambda} \rightarrow x = R\phi \rightarrow \ddot{x} = R\ddot{\phi} \rightarrow \ddot{\phi} = \ddot{x}/R$$

$$\lambda R = I(\ddot{x}/R) \rightarrow \lambda = \ddot{x} I/R^2$$

$$mg - \frac{\ddot{x} I}{R^2} = m\ddot{x} \longrightarrow \ddot{x}\left(1+\frac{I}{mR^2}\right) = g$$

$$\ddot{x} = \frac{g}{1+(I/mR^2)}$$

$$\lambda = \ddot{x} I/R^2 = \frac{gI/R^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} = mg\left(\frac{I}{I+mR^2}\right)$$

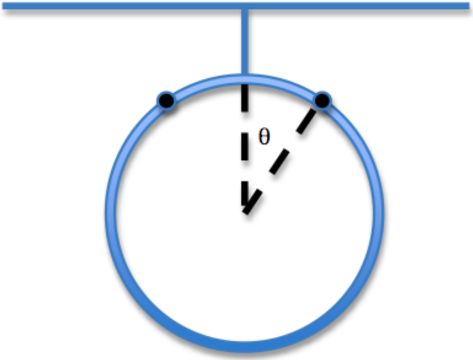
$$\lambda \frac{\partial f}{\partial x} = -mg\left(\frac{I}{I+mR^2}\right) \qquad \lambda \frac{\partial f}{\partial \phi} = +mgR\left(\frac{I}{I+mR^2}\right)$$

Newton: $ma = mg - F_T \rightarrow F_T = m(g-\ddot{x}) = mg - \dfrac{mg}{1+I/mR^2}$

$$F_T = mg\,\frac{1+(I/mR^2)-1}{1+I/mR^2} = mg\,\frac{I/mR^2}{1+I/mR^2} = \frac{mgI}{mR^2+I} \checkmark$$

A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction. The beads are released simultaneously from rest at the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > \frac{3}{2}M$, and find the angle θ at which this occurs. (If $M = 0$ then $\cos\theta = \frac{2}{3}$.)

Write $\mathcal{L}(\theta, \dot{\theta}, Y, \dot{Y})$ and include Lagrange multiplier term λY to enforce the $Y = 0$ constraint. "The ring starts to rise" implies $\lambda = 0$, i.e. string tension is zero.



```
In[ ]:= ClearAll["Global`*"];
y[t_] := Y[t] + R Cos[θ[t]];
x[t_] := R Sin[θ[t]];
T = m (x'[t]^2 + y'[t]^2) + M Y'[t]^2;
U = 2 m g y[t] + M g Y[t];
lag = T - U
```

$$-g\,M\,Y[t] - 2\,g\,m\,(R\,\mathrm{Cos}[\theta[t]] + Y[t]) + M\,Y'[t]^2 + m\left(R^2\,\mathrm{Cos}[\theta[t]]^2\,\theta'[t]^2 + (Y'[t] - R\,\mathrm{Sin}[\theta[t]]\,\theta'[t])^2\right)$$

```
In[ ]:= FullSimplify[lag]
```

$$-g\,(2\,m + M)\,Y[t] + (m + M)\,Y'[t]^2 - 2\,m\,R\,\mathrm{Sin}[\theta[t]]\,Y'[t]\,\theta'[t] + m\,R\left(-2\,g\,\mathrm{Cos}[\theta[t]] + R\,\theta'[t]^2\right)$$

```
]:= lagWithConstraint = lag + λ Y[t];
    FullSimplify[
     D[D[lagWithConstraint, Y'[t]], t] == D[lagWithConstraint, Y[t]]
    ]
```

```
8]= (2 m + M) (g + Y''[t]) == λ + 2 m R (Cos[θ[t]] θ'[t]^2 + Sin[θ[t]] θ''[t])
```

```
]:= FullSimplify[
     D[D[lagWithConstraint, θ'[t]], t] == D[lagWithConstraint, θ[t]]
    ]
```

```
9]= m R^2 θ''[t] == m R Sin[θ[t]] (g + Y''[t])
```

Now you can impose the constraint that $Y \equiv 0$, exploit the helpful fact that mechanical energy of each bead is constant (as long as $Y \equiv 0$), and solve for the condition that the force of constraint equals zero when the ring just barely starts to rise. You'll get a quadratic equation for $\cos\theta$ in terms of the ratio $3M/2m$.

$$(2m+M)g + 2(m+M)\ddot{Y} = \lambda + 2mR\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)$$

$$mR^2\ddot{\theta} = mR\sin\theta\,(g+\ddot{Y}) \Rightarrow \ddot{\theta} = \frac{g}{R}\sin\theta$$

energy: $\frac{1}{2}mR^2\dot{\theta}^2 = mgR(1-\cos\theta) \rightarrow \dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)$

$$(2m+M)g = 2mR\left(\frac{2g}{R}(1-\cos\theta)\cos\theta + \frac{g}{R}\sin\theta\sin\theta\right)$$

$$1+\frac{M}{2m} = 2\cos\theta - 2\cos^2\theta + (1-\cos^2\theta) = -3\cos^2\theta + 2\cos\theta + 1$$

$$3\cos^2\theta - 2\cos\theta + \frac{M}{2m} = 0$$

$$\cos\theta = \frac{1}{6}\left(2 \pm \sqrt{4-4(3)\left(\frac{M}{2m}\right)}\right) = \frac{1}{3}\left(1 \pm \sqrt{1-\frac{3M}{2m}}\right)$$

Non-uniqueness of the Lagrangian.

In classical mechanics, finding the path $\vec{x}(t)$ that makes $\mathcal{L}$ stationary is an elegant (and usually labor-saving) trick that helps us to find the EOM that $\vec{F} = m\vec{a}$ would have given us anyway.

Our only demand on $\mathcal{L}$ is that it give us the correct EOM.

If you happen to have one $\mathcal{L}$ that gives you the correct EOM, it is easy to find another: Consider

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}t} F(\vec{r}_\alpha, t)$$

We've added to $\mathcal{L}$ the total time derivative of a function $F(\vec{r}_\alpha, t)$, where F can be a function of the particles' positions and of time, but F **cannot** be a function of the particles' velocities.

Claim: $\mathcal{L}'$ gives the same EOM as $\mathcal{L}$.

To keep the notation simpler, use just two coordinates, x and y

$$\mathcal{L}'(x, \dot{x}, y, \dot{y}, t) = \mathcal{L}(x, \dot{x}, y, \dot{y}, t) + \frac{\mathrm{d}F(x, y, t)}{\mathrm{d}t}$$

The equations of motion for the original Lagrangian, $\mathcal{L}$, are

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

So the claim is that the same $x(t)$ and $y(t)$ the satisfy the above EOM also satisfy

$$\frac{\partial \mathcal{L}'}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}'}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}'}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}'}{\partial \dot{y}}$$

To prove this, we will show that for arbitrary $x(t)$ and $y(t)$,

$$\frac{\partial(\mathcal{L}' - \mathcal{L})}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L}' - \mathcal{L})}{\partial \dot{x}} \qquad \frac{\partial(\mathcal{L}' - \mathcal{L})}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L}' - \mathcal{L})}{\partial \dot{y}}$$

$$\mathcal{L}'(x,\dot{x},y,\dot{y},t) = \mathcal{L}(x,\dot{x},y,\dot{y},t) + \frac{\mathrm{d}F(x,y,t)}{\mathrm{d}t}$$

We want to show that ("LHS" $=$ "RHS"):

$$\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial \dot{x}}$$

Writing out the total derivative $\mathrm{d}F/\mathrm{d}t$:

$$\frac{\mathrm{d}F(x,y,t)}{\mathrm{d}t} = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial t}$$

Plugging this $\mathrm{d}F/\mathrm{d}t$ into the LHS:

$$\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial x} = \frac{\partial}{\partial x}\,\frac{\mathrm{d}F(x,y,t)}{\mathrm{d}t} = \frac{\partial}{\partial x}\,\frac{\partial F}{\partial x}\dot{x} + \frac{\partial}{\partial x}\,\frac{\partial F}{\partial y}\dot{y} + \frac{\partial}{\partial x}\,\frac{\partial F}{\partial t}$$

Then plugging same $\mathrm{d}F/\mathrm{d}t$ into the RHS:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial}{\partial \dot{x}}\left(\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial F}{\partial x}$$

Plugging this $\mathrm{d}F/\mathrm{d}t$ into the LHS:

$$\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial x} = \frac{\partial}{\partial x}\frac{\mathrm{d}F(x,y,t)}{\mathrm{d}t} = \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\dot{y} + \frac{\partial}{\partial x}\frac{\partial F}{\partial t}$$

Then plugging same $\mathrm{d}F/\mathrm{d}t$ into the RHS:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\left(\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial x}$$

then expanding the RHS using the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{x} + \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\dot{y} + \frac{\partial}{\partial t}\frac{\partial F}{\partial x}$$

and swapping the order of the partial derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\dot{y} + \frac{\partial}{\partial x}\frac{\partial F}{\partial t}$$

which is identical to the LHS above. So $\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L}'-\mathcal{L})}{\partial \dot{x}}$

Therefore, the Lagrangian

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}t} F(\vec{r}_\alpha, t)$$

has the same EOM as the original Lagrangian $\mathcal{L}$.

We've added to $\mathcal{L}$ the total time derivative of a function $F(\vec{r}_\alpha, t)$, where F can be a function of $\vec{r}_\alpha$ and of t, but F **cannot** be a function of the velocities $\dot{\vec{r}}_\alpha$.

This turns out to be **the most general** addition that you can make to $\mathcal{L}$ without affecting the equations of motion.

Another way to see this is to look at the action $\int \mathcal{L}\, \mathrm{d}t$

$$\int_{t_i}^{t_f} (\mathcal{L}' - \mathcal{L})\, \mathrm{d}t = \int_{t_i}^{t_f} \left(\frac{\mathrm{d}F(\vec{r}, t)}{\mathrm{d}t} \right) \mathrm{d}t = [F(\vec{r}, t)]_{t_i}^{t_f}$$

since the coordinates $\vec{r}$ are fixed at the endpoints, we've simply added a constant to the action, which does not change what path extremizes the action. Argument wouldn't work for $F(\dot{r})$.

Examples that don't change EOM: $\mathcal{L} \to \mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}t}F(x,t)$

$$\mathcal{L} \to \mathcal{L} + At \qquad F(x,t) = At^2/2$$

$$\mathcal{L} \to \mathcal{L} + Af(t) \qquad F(x,t) = A\int_{t'=0}^{t} f(t')\mathrm{d}t'$$

$$\mathcal{L} \to \mathcal{L} + A\dot{x} \qquad F(x,t) = Ax$$

$$\mathcal{L} \to \mathcal{L} + Ax\dot{x} \qquad F(x,t) = Ax^2/2$$

$$\mathcal{L} \to \mathcal{L} + Ax^n\dot{x} \qquad F(x,t) = Ax^{n+1}/(n+1)$$

Examples that **do** change EOM:

$$\mathcal{L} \to \mathcal{L} + Ax^n \quad (n \neq 0)$$

$$\mathcal{L} \to \mathcal{L} + A\dot{x}^2$$

When considering "symmetries of $\mathcal{L}$" you really mean that EOM is unchanged. If an operation changes $\mathcal{L}$ in a way that doesn't affect EOM, then the operation is still considered a symmetry of $\mathcal{L}$.

Monday's reading questions:

(1) Name several conserved quantities and the corresponding ignorable coordinates for the Kepler problem.

Monday's reading questions:

(1) Name several conserved quantities and the corresponding ignorable coordinates for the Kepler problem.

Once $\mathcal{L}$ is rewritten in terms of CM coordinate $\boldsymbol{R}$ and relative coordinate $\boldsymbol{r}$,

$$\mathcal{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(r)$$

we find $\partial\mathcal{L}/\partial\boldsymbol{R} = 0$, so $\boldsymbol{R}$ is ignorable, and the corresponding conserved quantity is $\partial\mathcal{L}/\partial\dot{\boldsymbol{R}} \equiv \boldsymbol{P}$, the system's total linear momentum.

Then once $\mathcal{L}$ is further reduced (because $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is constant, due to $\boldsymbol{L}$ conservation) to the planar form

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

we find another ignorable coordinate, ϕ, corresponding to conservation of angular momentum $L_z \equiv \ell$: $\ell = \mu r^2\dot{\phi} = \text{const.}$

Several people also pointed out that since time does not appear explicitly in $\mathcal{L}$, the total energy is conserved.

It's also an interesting fact (not mentioned by Taylor) that for an inverse-square central force (like Newtonian gravity), the "Laplace-Runge-Lenz vector" (a.k.a. "LRL vector") is a constant of the motion: $\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - G m_1 m_2 \mu \hat{\boldsymbol{r}}$, which basically points along the major axis of the ellipse.

`en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector`

Interestingly, this conserved quantity does not have a corresponding ignorable coordinate, so it's less well known than $\boldsymbol{P}$ and $\boldsymbol{L}$. In the Hamiltonian formalism, one can show that $\boldsymbol{A}$ is conserved (for a $1/r$ potential) by showing that $[\boldsymbol{A}, H] = 0$, where [] denotes the "Poisson bracket," which is the classical analogue of the "commutator" that you will see in quantum mechanics.

(This is pure digression!)

Consider two masses m_1 and m_2 connected by a spring:

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_1 - x_2)^2$$

No "ignorable" coordinates: $\frac{\partial \mathcal{L}}{\partial x_1} = -k(x_1 - x_2) \neq 0$, etc.

But with a clever choice of generalized coordinates, e.g. let

$$x \equiv x_1 - x_2 \qquad M \equiv m_1 + m_2 \qquad X \equiv \frac{m_1 x_1 + m_2 x_2}{M}$$

we can rewrite the same $\mathcal{L}$ as

$$\mathcal{L}(x, \dot{x}, \dot{X}) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} \left(\frac{m_1 m_2}{M} \right) \dot{x}^2 - \frac{1}{2} k x^2$$

where now X is "ignorable" ($\frac{\partial \mathcal{L}}{\partial X} = 0$) and the corresponding momemtum is a constant of the motion:

$$P \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}} = M \dot{X} = m_1 \dot{x}_1 + m_2 \dot{x}_2 = \mathrm{const.}$$

So one typically tries to choose generalized coordinates such that as many coordinates as possible are "ignorable," hence the corresponding momenta are conserved. Ch8 nicely illustrates this!

Verify that the positions of two particles can be written in terms of the CM and relative positions as

$$\boldsymbol{r}_1 = \boldsymbol{R} + m_2\boldsymbol{r}/M \qquad \boldsymbol{r}_2 = \boldsymbol{R} - m_1\boldsymbol{r}/M$$

where $M = m_1 + m_2$. Hence confirm that the total KE of the two particles can be expressed as

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2$$

where μ denotes the reduced mass $\mu = m_1 m_2/M$.

$$\underset{\sim}{r}_1 \overset{?}{=} \underset{\sim}{R} + \frac{m_2}{M}\underset{\sim}{r}$$

$$\frac{m_1 \underset{\sim}{r}_1 + m_2 \underset{\sim}{r}_2 + m_2(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)}{m_1 + m_2} = \underset{\sim}{r}_1$$

$$m_1 \dot{\underset{\sim}{r}}_1^2 = m_1 \left(\dot{\underset{\sim}{R}} + \frac{m_2}{M}\dot{\underset{\sim}{r}}\right)^2$$

$$= m_1 \dot{\underset{\sim}{R}}^2 + 2\frac{m_1 m_2}{M}\dot{\underset{\sim}{R}}\,\dot{\underset{\sim}{r}} + \frac{m_1 m_2^2}{M^2}\dot{\underset{\sim}{r}}^2$$

$$\underset{\sim}{r}_2 \overset{?}{=} \underset{\sim}{R} - \frac{m_1}{M}\underset{\sim}{r}$$

$$\frac{m_1 \underset{\sim}{r}_1 + m_2 \underset{\sim}{r}_2 - m_1(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)}{m_1 + m_2} = \underset{\sim}{r}_2$$

$$m_2 \dot{\underset{\sim}{r}}_2^2 = m_2 \left(\dot{\underset{\sim}{R}} - \frac{m_1}{M}\dot{\underset{\sim}{r}}\right)^2$$

$$= m_2 \dot{\underset{\sim}{R}}^2 - 2\frac{m_2 m_1}{M}\dot{\underset{\sim}{R}}\,\dot{\underset{\sim}{r}} + \frac{m_2 m_1^2}{M^2}\dot{\underset{\sim}{r}}^2$$

add $\Rightarrow (m_1 + m_2)\dot{\underset{\sim}{R}}^2 + \dfrac{(m_2 + m_1)m_1 m_2}{M^2}\dot{\underset{\sim}{r}}^2$

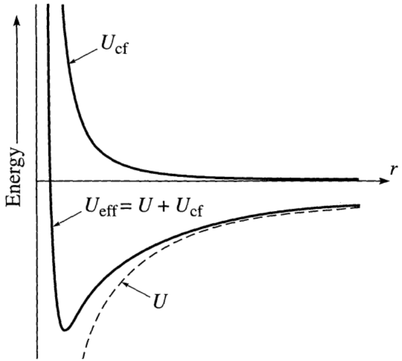
$$= M\dot{\underset{\sim}{R}}^2 + \frac{m_1 m_2}{M}\dot{\underset{\sim}{r}}^2 = M\dot{\underset{\sim}{R}}^2 + \mu\dot{\underset{\sim}{r}}^2$$

(2) Why is $U_{\rm eff}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$? Is the time for r to oscillate back and forth between $r_{\rm min}$ and $r_{\rm max}$ always equal to the time in which ϕ advances $360°$?

From $\partial\mathcal{L}/\partial r = \frac{\rm d}{{\rm d}t}(\partial\mathcal{L}/\partial\dot{r})$, we got the radial EOM,

$$\mu\ddot{r} \;=\; -\frac{{\rm d}U}{{\rm d}r} + \mu r\dot{\phi}^2 \;=\; -\frac{\rm d}{{\rm d}r}\left(-\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}\right)$$

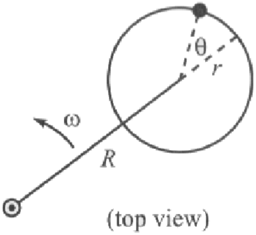
where "$\mu\omega^2 r$" centripetal term equals $\frac{\ell^2}{\mu r^3}$ since $\ell = \mu r^2\dot{\phi}$.



You can imagine r staying constant at bottom of "$U_{\rm eff}$," oscillating back and forth between $U_{\rm eff}(r_{\rm min}) = U_{\rm eff}(r_{\rm max})$, or else just bouncing/scattering once (bounded vs. unbounded).

$U_{\rm eff}$ = gravitational term + "centrifugal potential," which appears e.g. in HW4.q5 (wire on spinning horizontal hoop).

(future HW problem: generalization of HW4.q5)



A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point, as shown in the above-right figure. (a) Find the EOM for angle θ. (b) Find the frequency of small oscillations about the point of stable equilibrium.

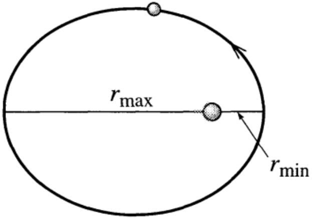
You'll find $\mathcal{L} = T$ $(U = 0)$, yet you still get oscillations about a stable equilibrium, due to ω-dependent centripetal terms that are somewhat analogous (but of a different form) to the "centrifugal potential" we find in the Kepler problem. You'll find

$$r\ddot{\theta} = -\omega^2 R \sin\theta$$

(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$?
Is the time for r to oscillate back and forth between $r_{\min}$ and $r_{\max}$ always equal to the time in which ϕ advances $360°$?

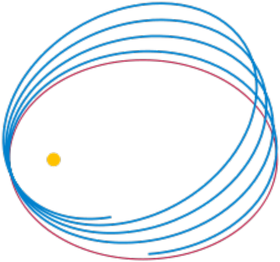
(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1 m_2/r$?
Is the time for r to oscillate back and forth between r_{min} and r_{max} always equal to the time in which ϕ advances $360°$?

For inverse-square-law forces $(U \sim -1/r)$ and for (isotropic) Hooke's-law forces $(U \sim r^2)$, the period of the ϕ motion equals the period of the r motion [actually $T_\phi = 2T_r$ for $(U \sim r^2)$], and the orbit always closes on itself after one revolution. For more general $U(r)$, the orbit does not necessarily repeat itself (non-closed orbit).



(a)

(inverse-square force)



(more general case)

(Taylor 8.12. Let's do this in class, today or next time.)

(a) By examining $\mathrm{d}/\mathrm{d}r$ of the radial effective potential

$$U_{\mathrm{eff}}(r) \;=\; -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

find the radius r_0 at which a planet with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius.

(b) Use $\mathrm{d}^2U_{\mathrm{eff}}/\mathrm{d}r^2$ to show that this circular orbit is stable, i.e. that a small radial nudge will cause only small radial oscillations.

(c) Show that the frequency Ω of these radial oscillations equals the frequency $\omega = \dot{\phi}$ of the planet's orbital motion.

$$U_{eff}(r) = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

(a)

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{G m_1 m_2}{r_0^2} - \frac{2\ell^2}{2\mu r_0^3} \implies r_0 = \frac{\ell^2}{G m_1 m_2 \mu}$$

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

(a)

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{Gm_1m_2}{r_0^2} - \frac{2\ell^2}{2\mu r_0^3} \Rightarrow r_0 = \frac{\ell^2}{Gm_1m_2\mu}$$

(b)

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = -\frac{2Gm_1m_2}{r_0^3} + \frac{3\ell^2}{\mu r_0^4} = \frac{1}{r_0^3}\left[-2Gm_1m_2 + \frac{3(Gm_1m_2\mu r_0)}{\mu r_0}\right]$$

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = +\frac{Gm_1m_2}{r_0^3} > 0$$

(stable equilibrium)

(c) $\mu \ddot{r} \approx - \left[\frac{d^2 U_{eff}}{dr^2} \right]_{r=r_0} (r - r_0)$

$\Omega^2 = \frac{U''_{eff}}{\mu} = \frac{G m_1 m_2}{\mu r_0^3}$ for small radial oscillations about r_0

(c) $$\mu \ddot{r} \approx -\left[\frac{d^2 U_{eff}}{dr^2}\right]_{r=r_0} (r - r_0)$$

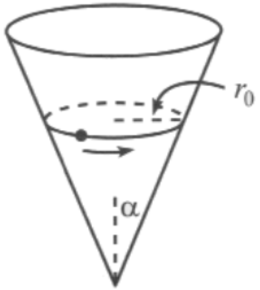
$$\Omega^2 = \frac{U''_{eff}}{\mu} = \frac{G m_1 m_2}{\mu r_0^3}$$ for small radial oscillations about r_0

Meanwhile, angular motion is given by equating centripetal force to gravitational force:

$$\mu \omega^2 r = \frac{G m_1 m_2}{r^2} \Rightarrow \omega^2 = \frac{G m_1 m_2}{\mu r_0^3}$$

So ω^2 (azimuthal) $= \Omega^2$ (radial)

A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



let $\rho = r_0 + \varepsilon \Rightarrow (1+c^2)\ddot{\rho} = \frac{\ell^2}{m^2(r_0+\varepsilon)^3} - gc$

$(1+c^2)\ddot{\varepsilon} = \frac{\ell^2}{m^2 r_0^3 (1+\frac{\varepsilon}{r_0})^3} - gc$

also note: $gc = \omega_0^2 r_0$
also $\ell = m r_0^2 \omega_0$

$(1+c^2)\ddot{\varepsilon} = \frac{\ell^2}{m^2 r_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right) - \omega_0^2 r_0$

$= \frac{(m r_0^2 \omega_0)^2}{m^2 r_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right) - \omega_0^2 r_0 = -\omega_0^2 r_0 \cdot \frac{3\varepsilon}{r_0}$

$$\ddot{\varepsilon} = -\boxed{\frac{3\omega_0^2}{(1+c^2)}}\,\varepsilon$$

Alternative: say $(1+c^2)\ddot{\rho} = f(\rho) \simeq f(r_0) + \varepsilon f'(r_0)$

$f(\rho) = \frac{\ell^2}{m^2\rho^3} - gc \qquad f(r_0) = \frac{\ell^2}{m^2 r_0^3} - gc = 0$

$f'(\rho) = -\frac{3\ell^2}{m^2\rho^4} \qquad f'(r_0) = -\frac{3\ell^2}{m^2 r_0^4}$

$\varepsilon f'(r_0) = -3\omega_0^2 \varepsilon$

Physics 351 — Wednesday, February 14, 2018

- HW4 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm.