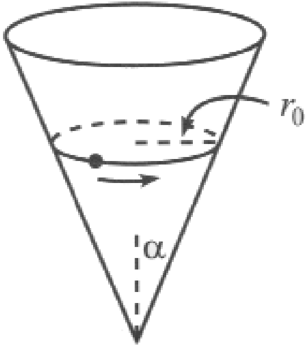
Physics 351 — Friday, February 16, 2018

- Turn in HW4. Pick up HW5 handout (due next Friday).
- Read Ch 9 (mechanics in non-inertial frames) this weekend, though it will be mid-week before we start discussing it.
- (The midterm exam (March 26) will cover chapters 7,8,9.)

A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



let $\rho = r_0 + \varepsilon \Rightarrow (1+c^2)\ddot{\rho} = \frac{\ell^2}{m^2(r_0+\varepsilon)^3} - gc$

$(1+c^2)\ddot{\varepsilon} = \frac{\ell^2}{m^2 r_0^3 (1+\frac{\varepsilon}{r_0})^3} - gc$

also note: $gc = \omega_0^2 r_0$
also $\ell = m r_0^2 \omega_0$

$(1+c^2)\ddot{\varepsilon} = \frac{\ell^2}{m^2 r_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right) - \omega_0^2 r_0$

$= \frac{(m r_0^2 \omega_0)^2}{m^2 r_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right) - \omega_0^2 r_0 = -\omega_0^2 r_0 \cdot \frac{3\varepsilon}{r_0}$

$\ddot{\varepsilon} = -\boxed{\frac{3\omega_0^2}{(1+c^2)}}\varepsilon$

Alternative: say $(1+c^2)\ddot{\rho} = f(\rho) \simeq f(r_0) + \varepsilon f'(r_0)$

$f(\rho) = \frac{\ell^2}{m^2\rho^3} - gc \qquad f(r_0) = \frac{\ell^2}{m^2 r_0^3} - gc = 0$

$f'(\rho) = -\frac{3\ell^2}{m^2\rho^4} \qquad f'(r_0) = -\frac{3\ell^2}{m^2 r_0^4}$

$\varepsilon f'(r_0) = -3\omega_0^2\varepsilon$

Last weekend's reading questions:

(1) Name several conserved quantities and the corresponding ignorable coordinates for the Kepler problem.

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Once $\mathcal{L}$ is rewritten in terms of CM coordinate $\boldsymbol{R}$ and relative coordinate $\boldsymbol{r}$,

$$\mathcal{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(r)$$

we find $\partial\mathcal{L}/\partial\boldsymbol{R} = 0$, so $\boldsymbol{R}$ is ignorable, and the corresponding conserved quantity is $\partial\mathcal{L}/\partial\dot{\boldsymbol{R}} \equiv \boldsymbol{P}$, the system's total linear momentum.

Then once $\mathcal{L}$ is further reduced (because $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is constant, due to $\boldsymbol{L}$ conservation) to the planar form

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

we find another ignorable coordinate, ϕ, corresponding to conservation of angular momentum $L_z \equiv \ell$: $\ell = \mu r^2\dot{\phi} = \text{const.}$

Several people also pointed out that since time does not appear explicitly in $\mathcal{L}$, the total energy is conserved.

It's also an interesting fact (not mentioned by Taylor) that for an inverse-square central force (like Newtonian gravity), the "Laplace-Runge-Lenz vector" (a.k.a. "LRL vector") is a constant of the motion: $\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - G m_1 m_2 \mu \hat{\boldsymbol{r}}$, which basically points along the major axis of the ellipse.

en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector

Interestingly, this conserved quantity does not have a corresponding ignorable coordinate, so it's less well known than $\boldsymbol{P}$ and $\boldsymbol{L}$. In the Hamiltonian formalism, one can show that $\boldsymbol{A}$ is conserved (for a $1/r$ potential) by showing that $[\boldsymbol{A}, H] = 0$, where $[]$ denotes the "Poisson bracket," which is the classical analogue of the "commutator" that you will see in quantum mechanics.

(This is pure digression!)

Consider two masses m_1 and m_2 connected by a spring:

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2$$

No "ignorable" coordinates: $\frac{\partial \mathcal{L}}{\partial x_1} = -k(x_1 - x_2) \neq 0$, etc.

But with a clever choice of generalized coordinates, e.g. let

$$x \equiv x_1 - x_2 \qquad M \equiv m_1 + m_2 \qquad X \equiv \frac{m_1 x_1 + m_2 x_2}{M}$$

we can rewrite the same $\mathcal{L}$ as

$$\mathcal{L}(x, \dot{x}, \dot{X}) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{M}\right)\dot{x}^2 - \frac{1}{2}kx^2$$

where now X is "ignorable" ($\frac{\partial \mathcal{L}}{\partial X} = 0$) and the corresponding momemtum is a constant of the motion:

$$P \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}} = M\dot{X} = m_1\dot{x}_1 + m_2\dot{x}_2 = \text{const.}$$

So one typically tries to choose generalized coordinates such that as many coordinates as possible are "ignorable," hence the corresponding momenta are conserved. Ch8 nicely illustrates this!

Verify that the positions of two particles can be written in terms of the CM and relative positions as

$$\boldsymbol{r}_1 = \boldsymbol{R} + m_2\boldsymbol{r}/M \qquad \boldsymbol{r}_2 = \boldsymbol{R} - m_1\boldsymbol{r}/M$$

where $M = m_1 + m_2$. Hence confirm that the total KE of the two particles can be expressed as

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2$$

where μ denotes the reduced mass $\mu = m_1m_2/M$.

$$\underset{\sim}{r_1} \stackrel{?}{=} \underset{\sim}{R} + \frac{m_2}{M}\underset{\sim}{r}$$

$$\frac{m_1 \underset{\sim}{r_1} + m_2 \underset{\sim}{r_2} + m_2(\underset{\sim}{r_1} - \underset{\sim}{r_2})}{m_1 + m_2} = \underset{\sim}{r_1}$$

$$m_1 \dot{\underset{\sim}{r_1}}^2 = m_1\left(\dot{\underset{\sim}{R}} + \frac{m_2}{M}\dot{\underset{\sim}{r}}\right)^2$$

$$= m_1 \dot{\underset{\sim}{R}}^2 + 2\frac{m_1 m_2}{M}\dot{\underset{\sim}{R}}\,\dot{\underset{\sim}{r}} + \frac{m_1 m_2^2}{M^2}\dot{\underset{\sim}{r}}^2$$

$$\underset{\sim}{r_2} \stackrel{?}{=} \underset{\sim}{R} - \frac{m_1}{M}\underset{\sim}{r}$$

$$\frac{m_1 \underset{\sim}{r_1} + m_2 \underset{\sim}{r_2} - m_1(\underset{\sim}{r_1} - \underset{\sim}{r_2})}{m_1 + m_2} = \underset{\sim}{r_2}$$

$$m_2 \dot{\underset{\sim}{r_2}}^2 = m_2\left(\dot{\underset{\sim}{R}} - \frac{m_1}{M}\dot{\underset{\sim}{r}}\right)^2$$

$$= m_2 \dot{\underset{\sim}{R}}^2 - 2\frac{m_2 m_1}{M}\dot{\underset{\sim}{R}}\,\dot{\underset{\sim}{r}} + \frac{m_2 m_1^2}{M^2}\dot{\underset{\sim}{r}}^2$$

add $\Rightarrow (m_1 + m_2)\dot{\underset{\sim}{R}}^2 + \dfrac{(m_2 + m_1)m_1 m_2}{M^2}\dot{\underset{\sim}{r}}^2$

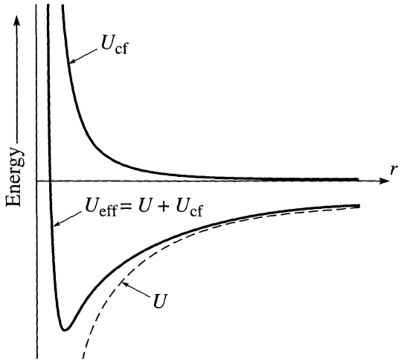
$$= M\dot{\underset{\sim}{R}}^2 + \frac{m_1 m_2}{M}\dot{\underset{\sim}{r}}^2 = M\dot{\underset{\sim}{R}}^2 + \mu\dot{\underset{\sim}{r}}^2$$

(2) Why is $U_{\rm eff}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$? Is the time for r to oscillate back and forth between $r_{\rm min}$ and $r_{\rm max}$ always equal to the time in which ϕ advances $360°$?

From $\partial\mathcal{L}/\partial r = \frac{\rm d}{{\rm d}t}(\partial\mathcal{L}/\partial\dot{r})$, we got the radial EOM,

$$\mu\ddot{r} \;=\; -\frac{{\rm d}U}{{\rm d}r} + \mu r\dot{\phi}^2 \;=\; -\frac{\rm d}{{\rm d}r}\left(-\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}\right)$$

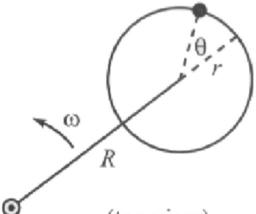
where "$\mu\omega^2 r$" centripetal term equals $\frac{\ell^2}{\mu r^3}$ since $\ell = \mu r^2\dot{\phi}$.



You can imagine r staying constant at bottom of "$U_{\rm eff}$," oscillating back and forth between $U_{\rm eff}(r_{\rm min}) = U_{\rm eff}(r_{\rm max})$, or else just bouncing/scattering once (bounded vs. unbounded).

$U_{\rm eff}$ = gravitational term + "centrifugal potential," which appears e.g. in HW4.q5 (wire on spinning horizontal hoop).

(future HW problem: generalization of HW4.q5)



A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point, as shown in the above-right figure. (a) Find the EOM for angle θ. (b) Find the frequency of small oscillations about the point of stable equilibrium.

You'll find $\mathcal{L} = T$ ($U = 0$), yet you still get oscillations about a stable equilibrium, due to ω-dependent centripetal terms that are somewhat analogous (but of a different form) to the "centrifugal potential" we find in the Kepler problem. You'll find

$$r\ddot{\theta} = -\omega^2 R \sin\theta$$

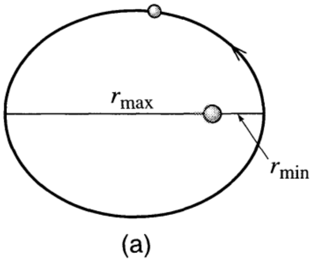
(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$?

Is the time for r to oscillate back and forth between $r_{\min}$ and $r_{\max}$ always equal to the time in which ϕ advances 360°?

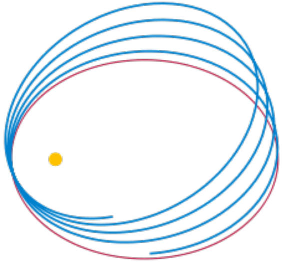
(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$?

Is the time for r to oscillate back and forth between $r_{\min}$ and $r_{\max}$ always equal to the time in which ϕ advances $360°$?

For inverse-square-law forces $(U \sim -1/r)$ and for (isotropic) Hooke's-law forces $(U \sim r^2)$, the period of the ϕ motion equals the period of the r motion [actually $T_\phi = 2T_r$ for $(U \sim r^2)$], and the orbit always closes on itself after one revolution. For more general $U(r)$, the orbit does not necessarily repeat itself (non-closed orbit).



(inverse-square force)



(more general case)

(Taylor 8.12. Let's do this in class, today or next time.)

(a) By examining $\mathrm{d}/\mathrm{d}r$ of the radial effective potential

$$U_{\text{eff}}(r) \;=\; -\,\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

find the radius r_0 at which a planet with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius.

(b) Use $\mathrm{d}^2U_{\text{eff}}/\mathrm{d}r^2$ to show that this circular orbit is stable, i.e. that a small radial nudge will cause only small radial oscillations.

(c) Show that the frequency Ω of these radial oscillations equals the frequency $\omega = \dot{\phi}$ of the planet's orbital motion.

$$U_{eff}(r) = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

(a)

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{G m_1 m_2}{r_0^2} - \frac{2\ell^2}{2\mu r_0^3} \Rightarrow r_0 = \frac{\ell^2}{G m_1 m_2 \mu}$$

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2}$$

(a)

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{Gm_1m_2}{r_0^2} - \frac{2l^2}{2\mu r_0^3} \Rightarrow r_0 = \frac{l^2}{Gm_1m_2\mu}$$

(b)

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = -\frac{2Gm_1m_2}{r_0^3} + \frac{3l^2}{\mu r_0^4} = \frac{1}{r_0^3}\left[-2Gm_1m_2 + \frac{3(Gm_1m_2\mu r_0)}{\mu r_0}\right]$$

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = +\frac{Gm_1m_2}{r_0^3} > 0$$

(stable equilibrium)

(c) $\mu \ddot{r} \approx - \left[\frac{d^2 U_{eff}}{dr^2} \right]_{r=r_0} (r - r_0)$

$\Omega^2 = \frac{U''_{eff}}{\mu} = \frac{G m_1 m_2}{\mu r_0^3}$ for small radial oscillations about r_0

(c) $$\mu \ddot{r} \approx -\left[\frac{d^2 U_{eff}}{dr^2}\right]_{r=r_0} (r - r_0)$$

$$\Omega^2 = \frac{U''_{eff}}{\mu} = \frac{G m_1 m_2}{\mu r_0^3}$$

for small radial oscillations about r_0.

Meanwhile, angular motion is given by equating centripetal force to gravitational force:

$$\mu \omega^2 r = \frac{G m_1 m_2}{r^2} \Rightarrow \omega^2 = \frac{G m_1 m_2}{\mu r_0^3}$$

So ω^2 (azimuthal) $= \Omega^2$ (radial)

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