Physics 351 — Monday, February 19, 2018

- You read Ch 9 (mechanics in non-inertial frames) last weekend, though it will be mid-week before we start discussing it.
- (The midterm exam (March 26) will cover chapters 7,8,9.)

(a) By examining $\mathrm{d}/\mathrm{d}r$ of the radial effective potential

$$U_{\text{eff}}(r) \;=\; -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

find the radius r_0 at which a planet with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius.

(b) Use $\mathrm{d}^2U_{\text{eff}}/\mathrm{d}r^2$ to show that this circular orbit is stable, i.e. that a small radial nudge will cause only small radial oscillations.

(c) Show that the frequency Ω of these radial oscillations equals the frequency $\omega = \dot{\phi}$ of the planet's orbital motion.

This was the last thing we did on Friday

Consider two masses m_1 and m_2 connected by a spring:

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2$$

No "ignorable" coordinates: $\frac{\partial \mathcal{L}}{\partial x_1} = -k(x_1 - x_2) \neq 0$, etc.

But with a clever choice of generalized coordinates, e.g. let

$$x \equiv x_1 - x_2 \qquad M \equiv m_1 + m_2 \qquad X \equiv \frac{m_1 x_1 + m_2 x_2}{M}$$

we can rewrite the same $\mathcal{L}$ as

$$\mathcal{L}(x, \dot{x}, \dot{X}) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{M}\right)\dot{x}^2 - \frac{1}{2}kx^2$$

where now X is "ignorable" ($\frac{\partial \mathcal{L}}{\partial X} = 0$) and the corresponding momemtum is a constant of the motion:

$$P \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}} = M\dot{X} = m_1\dot{x}_1 + m_2\dot{x}_2 = \text{const.}$$

So one typically tries to choose generalized coordinates such that as many coordinates as possible are "ignorable."

(You'll do this on HW5, so let's not do it in class.)

Verify that the positions of two particles can be written in terms of the CM and relative positions as

$$\boldsymbol{r}_1 = \boldsymbol{R} + m_2\boldsymbol{r}/M \qquad \boldsymbol{r}_2 = \boldsymbol{R} - m_1\boldsymbol{r}/M$$

where $M = m_1 + m_2$. Hence confirm that the total KE of the two particles can be expressed as

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2$$

where μ denotes the reduced mass $\mu = m_1 m_2/M$.

$$\mathbf{r}_1 \stackrel{?}{=} \mathbf{R} + \frac{m_2}{M}\mathbf{r}$$

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_2(\mathbf{r}_1 - \mathbf{r}_2)}{m_1+m_2} = \mathbf{r}_1$$

$$m_1\dot{\mathbf{r}}_1^2 = m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2$$

$$= m_1\dot{\mathbf{R}}^2 + 2\frac{m_1 m_2}{M}\dot{\mathbf{R}}\dot{\mathbf{r}}$$

$$+ \frac{m_1 m_2^2}{M^2}\dot{\mathbf{r}}^2$$

$$\mathbf{r}_2 \stackrel{?}{=} \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 - m_1(\mathbf{r}_1 - \mathbf{r}_2)}{m_1+m_2} = \mathbf{r}_2$$

$$m_2\dot{\mathbf{r}}_2^2 = m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2$$

$$= m_2\dot{\mathbf{R}}^2 - 2\frac{m_2 m_1}{M}\dot{\mathbf{R}}\dot{\mathbf{r}}$$

$$+ \frac{m_2 m_1^2}{M^2}\dot{\mathbf{r}}^2$$

add $\Rightarrow (m_1+m_2)\dot{\mathbf{R}}^2 + \frac{(m_2+m_1)m_1 m_2}{M^2}\dot{\mathbf{r}}^2$

$$= M\dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M}\dot{\mathbf{r}}^2 = M\dot{\mathbf{R}}^2 + \mu\dot{\mathbf{r}}^2$$

In the Kepler problem, we started out with

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \frac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$$

Then using CM coordinate $\boldsymbol{R}$ (and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$) let us write

$$\mathcal{L} = \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r)$$

making $\boldsymbol{R}$ ignorable. Working in (inertial) CM frame let us write

$$\mathcal{L}_{\rm rel} = \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r)$$

Angular-momentum conservation keeps motion in fixed plane, so we can work in 2D polar coordinates:

$$\mathcal{L} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

making ϕ ignorable: the ϕ EOM is just

$$\mu r^2 \dot{\phi} \equiv \ell$$

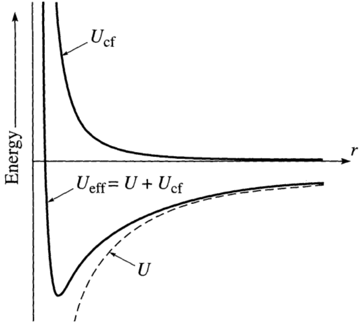
$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r) \qquad\qquad \mu r^2\dot{\phi} \equiv \ell$$

Radial EOM is (you **cannot** plug in $\dot{\phi} = \ell/(\mu r^2)$ before this):

$$\mu\ddot{r} = -\frac{\mathrm{d}U}{\mathrm{d}r} + \mu r\dot{\phi}^2 = -\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{\ell^2}{\mu r^3} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(U(r) + \frac{\ell^2}{2\mu r^2}\right)$$

where the "effective potential" is U + "centrifugal potential"

$$U_{\text{eff}}(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$



Now energy conservation lets us write

$$E \equiv \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

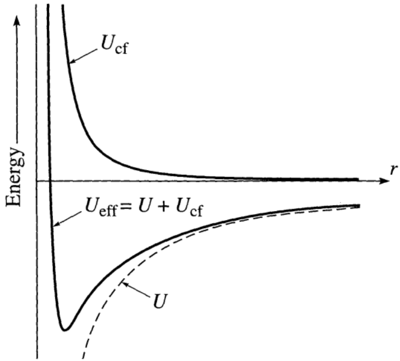
So r just oscillates back and forth at constant E.

(2) Why is $U_{\rm eff}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$? Is the time for r to oscillate back and forth between $r_{\rm min}$ and $r_{\rm max}$ always equal to the time in which ϕ advances $360°$?

From $\partial\mathcal{L}/\partial r = \frac{\mathrm{d}}{\mathrm{d}t}(\partial\mathcal{L}/\partial\dot{r})$, we got the radial EOM,

$$\mu\ddot{r} \;=\; -\frac{\mathrm{d}U}{\mathrm{d}r} + \mu r\dot{\phi}^2 \;=\; -\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}\right)$$

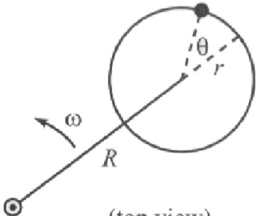
where "$\mu\omega^2 r$" centripetal term equals $\frac{\ell^2}{\mu r^3}$ since $\ell = \mu r^2\dot{\phi}$.



You can imagine r staying constant at bottom of "$U_{\rm eff}$," oscillating back and forth between $U_{\rm eff}(r_{\rm min}) = U_{\rm eff}(r_{\rm max})$, or else just bouncing/scattering once (bounded vs. unbounded).

$U_{\rm eff}$ = gravitational term + "centrifugal potential," which appeared e.g. in HW4.q5 (wire on spinning horizontal hoop).

(HW5 problem: generalization of HW4.q5)



A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point, as shown in the above-right figure. (a) Find the EOM for angle θ. (b) Find the frequency of small oscillations about the point of stable equilibrium.

You'll find $\mathcal{L} = T$ $(U = 0)$, yet you still get oscillations about a stable equilibrium, due to ω-dependent centripetal terms that are somewhat analogous (but of a different form) to the "centrifugal potential" we find in the Kepler problem. You'll find

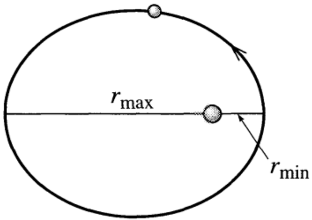
$$r\ddot{\theta} = -\omega^2 R \sin\theta$$

(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$?

Is the time for r to oscillate back and forth between $r_{\min}$ and $r_{\max}$ always equal to the time in which ϕ advances 360°?

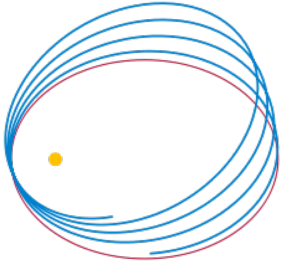
(2) Why is $U_{\text{eff}}(r)$ non-monotonic, unlike $U(r) = -Gm_1m_2/r$?
Is the time for r to oscillate back and forth between $r_{\min}$ and $r_{\max}$ always equal to the time in which ϕ advances $360°$?

For inverse-square-law forces $(U \sim -1/r)$ and for (isotropic) Hooke's-law forces $(U \sim r^2)$, the period of the ϕ motion equals the period of the r motion [actually $T_\phi = 2T_r$ for $(U \sim r^2)$], and the orbit always closes on itself after one revolution. For more general $U(r)$, the orbit does not necessarily repeat itself (non-closed orbit).



(a)

(inverse-square force)



(more general case)

(Taylor 8.12. Let's do this in class. You started it before class.)

(a) By examining $\mathrm{d}/\mathrm{d}r$ of the radial effective potential

$$U_{\text{eff}}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

find the radius r_0 at which a planet with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius.

(b) Use $\mathrm{d}^2U_{\text{eff}}/\mathrm{d}r^2$ to show that this circular orbit is stable, i.e. that a small radial nudge will cause only small radial oscillations.

(c) Show that the frequency Ω of these radial oscillations equals the frequency $\omega = \dot{\phi}$ of the planet's orbital motion.

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

ⓐ

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{Gm_1m_2}{r_0^2} - \frac{2\ell^2}{2\mu r_0^3} \Rightarrow r_0 = \frac{\ell^2}{Gm_1m_2\mu}$$

ⓑ

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = -\frac{2Gm_1m_2}{r_0^3} + \frac{3\ell^2}{\mu r_0^4} = \frac{1}{r_0^3}\left[-2Gm_1m_2 + \frac{3(Gm_1m_2\mu r_0)}{\mu r_0}\right]$$

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = +\frac{Gm_1m_2}{r_0^3} > 0$$

(stable equilibrium)

(c) $$\mu \ddot{r} \approx - \left[\frac{d^2 U_{eff}}{dr^2}\right]_{r=r_0} (r - r_0)$$

$$\Omega^2 = \frac{U''_{eff}}{\mu} = \frac{G m_1 m_2}{\mu r_0^3}$$ for small radial oscillations about r_0

Meanwhile, angular motion is given by equating centripetal force to gravitational force:

$$\mu \omega^2 r = \frac{G m_1 m_2}{r^2} \Rightarrow \omega^2 = \frac{G m_1 m_2}{\mu r_0^3}$$

So ω^2(azimuthal) $= \Omega^2$ (radial)

Radial equation of motion:

$$\mu\ddot{r} = -\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{\ell^2}{\mu r^3} = F(r) + \frac{\ell^2}{\mu r^3}$$

The ingenious trick to solving the radial EOM is to substitute $u = 1/r$.

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2\, u(\phi)^2}\, F$$

(which you'll use once or twice in HW6 next week).

If you plug in $F = 0$ (no force), you get $\quad u''(\phi) = -u(\phi) \quad$ whose solution is that u is a cosine (so then $1/r$ is a cosine):

$$r(\phi) = \frac{1}{u(\phi)} = \frac{r_0}{\cos(\phi - \delta)}$$

which (believe it or not!) is an equation for a straight line: $r\cos(\phi + \alpha) = \text{const.}$

Back to the "transformed radial equation"

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2\, u(\phi)^2}\, F$$

If we specialize to (attractive) inverse-square forces ($\gamma = Gm_1m_2$):

$$F(r) = -\frac{Gm_1m_2}{r^2} = -\frac{\gamma}{r^2} = -\gamma u^2$$

then the u^2 cancels, making the last term a constant

$$u''(\phi) = -u(\phi) - \frac{\gamma\mu}{\ell^2}$$

The solution to $(u + \mathrm{const})'' = -(u + \mathrm{const})$ is that $(u + \mathrm{const})$ is a cosine (with phase shift we'll take to be zero):

$$u(\phi) = \frac{\gamma\mu}{\ell^2}(1 + \epsilon\cos\phi)$$

$$r(\phi) = \frac{1}{u(\phi)} = \frac{\ell^2/(\gamma\mu)}{1 + \epsilon\cos\phi} = \frac{\ell^2/(Gm_1m_2\mu)}{1 + \epsilon\cos\phi} \equiv \frac{c}{1 + \epsilon\cos\phi}$$

So for the attractive inverse-square case $F = Gm_1 m_2 / r^2$ we find

$$r(\phi) = \frac{\ell^2/(Gm_1 m_2 \mu)}{1 + \epsilon \cos\phi} \equiv \frac{c}{1 + \epsilon \cos\phi}$$

where c has dimensions of length and the dimensionless parameter ϵ is the **eccentricity**.

As you know from the reading, this equation can describe a circle, an ellipse, a parabola, or a hyperbola.

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

(Taylor 8.19) The height of a satellite at perigee is 300 km above the earth's surface, and it is 3000 km at apogee.

(a) Find the orbit's eccentricity.

(b) If the orbit lies in the xy plane, with major axis in the x direction, and with earth's center at the origin, what is the satellite's height when it crosses the y axis?

Note: earth's radius is $R_e = 6400 \text{ km}$.

(At what value of $\cos \phi$ is r a minimum? A maximum?)

```
re = 6400.;
rmax = re + 3000;
rmin = re + 300;
ϵ = (rmax - rmin) / (rmax + rmin);
c = (1 + ϵ) rmin;
r[ϕ_] := c / (1 + ϵ Cos[ϕ]);
PolarPlot[r[ϕ], {ϕ, 0, 2 Pi},
 GridLines → Automatic]
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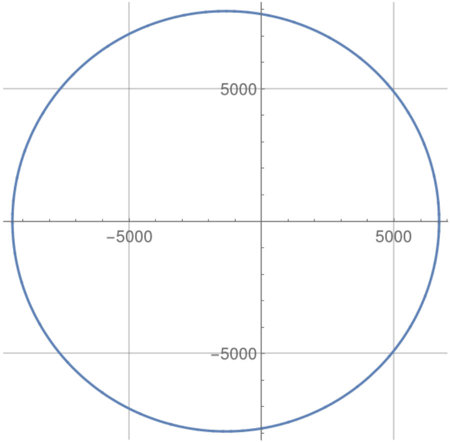

$$\frac{c}{1+\varepsilon} = 6700\,\text{km} \quad \Big\} \quad \frac{c}{1-\varepsilon} = 9400\,\text{km}$$

$$c = (1+\varepsilon) r_{min} = (1-\varepsilon) r_{max}$$

$$r_{max} - r_{min} = \varepsilon (r_{max} + r_{min})$$

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{9400 - 6700}{9400 + 6700} = 0.168$$

$$c = (1+\varepsilon) r_{min} = (1.168)(6700\,\text{km}) =$$
$$c = 7824\,\text{km}$$

$$y(x=0) = y(\cos\phi = 0) = c$$
$$= 7824\,\text{km}$$

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