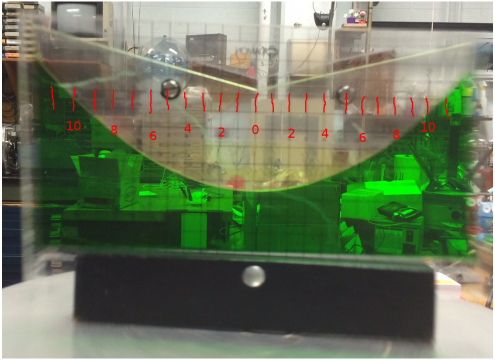
Physics 351 — Friday, February 23, 2018

- You'll read the first half (§10.1–10.7) of Chapter 10 this weekend. We'll start talking about Ch10 mid/late next week.
- The midterm exam (**March** 26) will cover (only!) chapters 7,8,9. But we will be working on Ch 10 by then.
- Turn in HW5. Pick up handout for HW6, due next Friday.



Argue why $y \propto r^2$ in Wednesday's spinning-liquid demo.

Gravity Well (Kepler's First Law of Planetary Motion) Hong Kong Space Museum

https://www.youtube.com/watch?v=4ffI0l8vUK8&t=12

This sort of resembles the scenario from problem 3.

For spinning Earth, $\boldsymbol{\Omega}$ points up out of the north pole. This rotation gives position $\vec{r}$ on Earth's surface the velocity

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$$

That is, as seen from space,

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{r}$$

This also works for a unit vector $\boldsymbol{e}$ fixed on the spinning globe:

$$\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{e}$$

At any given instant, you can project a given vector $\boldsymbol{Q}$ onto two different sets of axes: the "space" axes $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ that are **not** rotating (they're fixed in space somewhere), and the "body" axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ that are rotating with the globe (a.k.a. the rigid body).

If you observe **the same vector $\boldsymbol{Q}$** from two different frames:

- A "space" frame $\mathcal{S}_0$ that is stationary in space.
- A "body" frame $\mathcal{S}$ that is attached to the spinning globe.

$$\boldsymbol{Q} = Q_x \hat{\boldsymbol{x}} + Q_y \hat{\boldsymbol{y}} + Q_z \hat{\boldsymbol{z}} = Q_1 \hat{\boldsymbol{e}}_1 + Q_2 \hat{\boldsymbol{e}}_2 + Q_3 \hat{\boldsymbol{e}}_3$$

Then we use the product rule for the RHS to account for

$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_1}{\mathrm{d}t} = \boldsymbol{\Omega} \times \hat{\boldsymbol{e}}_1$$

which gives this incredibly useful identity:

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}_0(\text{space})} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}(\text{body})} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

You'll use this often to relate $\mathrm{d}\boldsymbol{Q}/\mathrm{d}t$ as observed in the two frames.

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}_0} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

Applying this (twice) to Newton's second law (inertial frame $\mathcal{S}_0$):

$$m\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}_0} = \boldsymbol{F}$$

gives Newton's 2nd law as seen in rotating frame $\mathcal{S}$

$$\boxed{m\ddot{\boldsymbol{r}} \;=\; \boldsymbol{F} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}}$$

and two well-known pseudo-forces appear. (This assumes $\dot{\boldsymbol{\Omega}} = 0$.)

2nd term is "Coriolis force." 3rd term is "centrifugal force."

(If $\dot{\boldsymbol{\Omega}} \neq 0$, then you get one more term, $m\boldsymbol{r} \times \dot{\boldsymbol{\Omega}}$, called the "azimuthal force" or "Euler force." See HW06/q9.)

Pause for two points: how to evaluate each quantity in the boxed equation; and which directions all the various vectors point.

The **centrifugal force** (which our high-school physics teachers regarded with abomination) is

$$\boldsymbol{F}_{\text{centrifugal}} = m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

(Use right-hand rule to see the various vectors on globe.)

Its magnitude is ... ?

The **centrifugal force** (which our high-school physics teachers regarded with abomination) is

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(Use right-hand rule to see the various vectors on globe.)

Its magnitude is ... ?

This is easier to see using $\rho = r \sin\theta$:

$$\boldsymbol{F}_{\text{centrifugal}} \;=\; m\Omega^2 \, \rho \, \hat{\boldsymbol{\rho}} \;=\; m\Omega^2 r \sin\theta \, \hat{\boldsymbol{\rho}}$$

Unlike the inertial force and the centrifugal force, which you experience when traveling in cars, airplanes, etc., the **Coriolis force** is much less familiar from everyday experience:

$$\boldsymbol{F}_{\mathrm{coriolis}} = 2m\,\boldsymbol{v} \times \boldsymbol{\Omega}$$

(Use right-hand rule to see the various vectors on globe.) One handy mnemonic is the analogy with the magnetic force law.

One more HW problem from a past year: What are the directions of the centrifugal and Coriolis forces on a person moving

(1) south near the North Pole?

(2) east on the equator?

(3) west on the equator?

(4) south across the equator?

(5) east near Philadelphia?

(6) west near Philadelphia?

(7) north near Philadelphia?

If you look down at Earth from space (e.g. from above the equator), and you somehow manage to hover at rest w.r.t. Earth's CoM, as if you were watching a spinning globe, in which direction do you see the land beneath you moving?

(A) toward the east (from west to east)

(B) toward the west (from east to west)

(C) It depends on whether you're looking down at the northern or the southern hemisphere.

In the same scenario as the last page (hovering as if watching a spinning globe), if I look down at Earth, then blink my eyes for a moment, then look down again at Earth, the land that I see after blinking is

(A) east of the land that I saw before blinking (I see Philly, then a while later I see England)

(B) west of the land that I saw before blinking (I see Philly, then a while later I see San Francisco)

If I drop an object straight down from the top of a tall building, the Coriolis force will deflect the falling object

(A) toward the east

(B) toward the west

$$\boldsymbol{F}_{\text{coriolis}} = 2m\,\boldsymbol{v} \times \boldsymbol{\Omega}$$

The cross-product of (downward toward Earth's center) with $\boldsymbol{\Omega}$ points

(A) toward the east

(B) toward the west

Let's work out together the Coriolis deflection of an object dropped (from rest) from a height h above Earth's surface. For simplicity, let the starting point be directly above the equator.

(We stopped here on Friday.)

$\boxed{\text{drop}}$ object from height h above equator

$$\underset{\sim}{F}_{coriolis} = 2m\,\underset{\sim}{v} \times \underset{\sim}{\Omega}$$

let $\hat{x} = $ east $\quad \hat{y} = $ north, $\quad \hat{z} = $ up

at equator, $\underset{\sim}{\Omega} = \Omega\,\hat{y}$

$$F_{coriolis} = 2m\left(v_x \Omega\,\hat{z} - v_z \Omega\,\hat{x}\right)$$

$$m\ddot{x} = -2m v_z \Omega$$

$$\ddot{x} = -2(-gt)\Omega$$
$$= 2g\Omega t$$

0th order: $v_z = -gt$

$$t_f = \sqrt{\frac{2h}{g}}$$

$$\dot{x} = g\Omega t^2$$

$$x_f = \frac{1}{3} g\Omega t^3 = \frac{1}{3} g\Omega\left(\frac{2h}{g}\right)^{3/2} = \boxed{\frac{2}{3}\Omega h\sqrt{\frac{2h}{g}}}$$

speed of zero) gives $v_{\text{east}} = \omega g t^2 \sin\theta$. Integrating again to obtain the eastward deflection (with an initial eastward deflection of zero) gives $d_{\text{east}} = \omega g t^3 \sin\theta/3$. Plugging in $t = \sqrt{2h/g}$ gives

$$d_{\text{east}} = \frac{2\omega h \sin\theta}{3}\sqrt{\frac{2h}{g}}. \tag{10.18}$$

The frequency of the earth's rotation is $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$, so if we pick $\theta = \pi/2$ and $h = 100$ m, for example, then we have $d_{\text{east}} \approx 2$ cm.

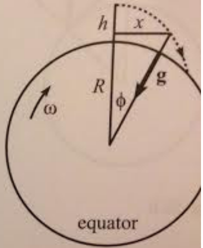
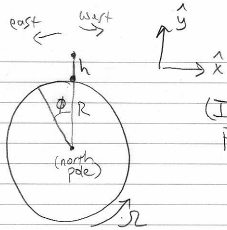


Fig. 10.8

REMARK: We can also solve this problem by working in an inertial frame; see Stirling (1983). Figure 10.8 shows the setup where a ball is dropped from a tower of height h located at the equator (the view is from the south pole). The earth is rotating in the inertial frame, so the initial sideways speed of the ball, $(R+h)\omega$, is larger than the sideways speed of the base of the tower, $R\omega$. This is the basic cause of the eastward deflection.

However, after the ball has moved to the right, the gravitational force on it picks up a component pointing to the left, and this slows down the sideways speed. If the ball has moved a distance x to the right, then the leftward component of gravity equals $g\sin\phi \approx g(x/R)$. Now, to leading order we have $x = R\omega t$, so the sideways acceleration of the ball is $a = -g(R\omega t/R) = -\omega g t$. Integrating this, and using the initial speed of $(R+h)\omega$, gives a rightward speed of $(R+h)\omega - \omega g t^2/2$. Integrating again gives a rightward distance of $(R+h)\omega t - \omega g t^3/6$. Subtracting off the rightward position of the base of the tower (namely $R\omega t$), and using $t \approx \sqrt{2h/g}$ (neglecting higher-order effects such as the curvature of the earth and the variation of g with altitude), we obtain an eastward deflection of $\omega h\sqrt{2h/g}(1 - 1/3) = (2/3)\omega h\sqrt{2h/g}$, relative to the base of the tower. If the ball is dropped at a polar angle θ instead of at the equator, then the only modification is that all speeds are decreased by a factor of $\sin\theta$, so we obtain the result in Eq. (10.18). ♣

Here is the same problem worked out from the inertial frame, instead of the Earth frame. I don't think it's worth going through in class, but here it is. Note that the sketch is looking down from the north pole.



(I chose $\hat{x}$ to point west here, confusingly.)

$$\underset{\sim}{F}_{grav} = -mg\hat{r} = -mg(\hat{y}\cos\Omega t - \hat{x}\sin\Omega t)$$

$$\simeq -mg\hat{y} + mg\Omega t\,\hat{x}$$

$$\ddot{y} = -g \qquad \ddot{x} = \Omega g t$$

$$\dot{x} = v_{x0} + \tfrac{1}{2}\Omega g t^2$$

$$x = v_{x0}t + \tfrac{1}{6}\Omega g t^3$$

$$= -(R+h)\Omega t + \tfrac{1}{6}\Omega g t^3$$

But bottom of tower has moved by

$$\Delta x = -R\Omega t$$

So (to first order in Ω) deflection of dropped object w.r.t. earth's surface is eastward by

$$h\Omega t - \tfrac{1}{6}\Omega g t^3 \qquad \left(t \simeq \sqrt{\tfrac{2h}{g}}\right)$$

$$= \left(\Omega\sqrt{\tfrac{2h}{g}}\right)\left(h - \tfrac{1}{6}g\left(\tfrac{2h}{g}\right)\right) = \tfrac{2}{3}\Omega h\sqrt{\tfrac{2h}{g}}$$

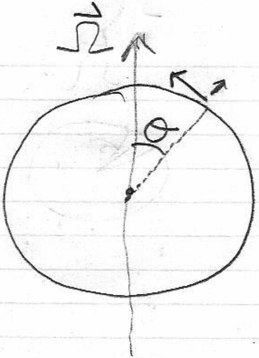
$$\boldsymbol{F}_{\text{coriolis}} = 2m\,\boldsymbol{v} \times \boldsymbol{\Omega}$$

A projectile of mass m is fired with initial speed v_0 horizontally and due north from a position of colatitude θ. Find the direction and magnitude of the Coriolis force in terms of m, v_0, θ, and Earth's angular velocity Ω.

How does the Coriolis force compare with the projectile's weight if $v_0 = 1000\ \text{m/s}$ and $\theta = 60°$? $\quad (\Omega \approx 7.3 \times 10^{-5}/\text{s}, \quad g \approx 10\ \text{m/s}^2)$

(1000 m/s is around Mach 3, i.e. $3\times$ the speed of sound. In more familiar units, it's 3600 kph, or 2240 mph. Wikipedia says this is the right order of magnitude for modern artillery cannons. I imagine (just guessing) that the Napoleonic-era cannon that severely wounded Prince Andrei in *War and Peace* was sub-sonic.)

$$\vec{F}_{coriolis} = 2m\vec{v} \times \vec{\Omega}$$



$$\vec{\Omega} = \Omega \sin\theta\,(\hat{north}) + \Omega\cos\theta\,(\hat{up})$$

$$(\hat{north}) \times (\hat{north}) = 0$$

$$(\hat{north}) \times (\hat{up}) = (\hat{east})$$

$$\Rightarrow \vec{F}_{\substack{coriolis \\ (initial)}} = 2mv_0\Omega\cos\theta\,(\hat{east})$$

if $\theta = 60°$ then $\cos\theta = \frac{1}{2}$ $\quad \Omega \approx \frac{2\pi}{86400} = 7.3\times10^{-5}/s$

$$\frac{2mv_0\Omega\cos\theta}{mg} = \frac{v_0\Omega}{g} \simeq \frac{(1000)(7.3\times10^{-5})}{10} = .007$$

(somewhat under 1% of mg)

From spring 2015 midterm: At a polar angle θ (colatitude), a projectile is fired due north with initial velocity v_0 at an inclination angle α above the ground. Working to first order in Earth's rotational velocity Ω, show that the eastward deflection due to the Coriolis force is (vs. time t since the projectile was fired)

$$x(t) = \Omega v_0 (\cos\alpha \cos\theta - \sin\alpha \sin\theta) t^2 + \frac{1}{3} \Omega g t^3 \sin\theta$$

[Assume that air resistance is negligible and that $\boldsymbol{g}$ is a constant throughout the flight. I recommend first neglecting the Coriolis force and writing the "zeroth order" $y(t)$ (northward) and $z(t)$ (upward) for ordinary projectile motion; then use this zeroth-order trajectory to calculate the first-order Coriolis deflection, after evaluating $\boldsymbol{\Omega}$ in terms of the local $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ axes.]

$$\ddot{x} = -2v_0\Omega\sin\theta\sin\alpha + 2(\Omega\sin\theta)gt$$

$$v_x = v_0\cos\alpha - 2(v_0\Omega\sin\theta\sin\alpha)t + (\Omega\sin\theta)gt^2$$

$$x = (v_0\cos\alpha)t - (v_0\Omega\sin\theta\sin\alpha)t^2 + \frac{1}{3}(\Omega\sin\theta)gt^3$$

$$x_f^{(1\text{st order})} = (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha}{g} + \frac{4v_0^2\Omega\sin\theta\sin\alpha\cos\alpha}{g^2}\right)$$

$$-(v_0\Omega\sin\theta\sin\alpha)\left(\frac{2v_0\sin\alpha}{g}\right)^2$$

$$+\frac{1}{3}(\Omega\sin\theta)g\left(\frac{2v_0\sin\alpha}{g}\right)^3$$

$$-\frac{12}{3} + \frac{8}{3} = -\frac{4}{3}$$

$$= \frac{2v_0^2\sin\alpha\cos\alpha}{g} + \frac{4v_0^3\Omega}{g^2}\sin\theta\sin\alpha\cos^2\alpha + \frac{v_0^3\Omega}{g^2}\sin\theta\sin^3\alpha\underbrace{\left(-4 + \frac{8}{3}\right)}$$

$$\Delta x \equiv x_f^{(1)} - x_f^{(0)} = \frac{4v_0^3\Omega}{g^2}\sin\theta\left(\sin\alpha\cos^2\alpha - \frac{1}{3}\sin^3\alpha\right)$$

A puck slides with speed v on frictionless ice. The surface is "level" in the sense that it is orthogonal to the effective (gravitational + centrifugal) $\boldsymbol{g}$ at all points. Show that the puck moves in a circle, as seen in Earth's rotating frame. (Assume that v is small enough that the radius of the circle is much smaller than the radius of Earth, so that the colatitude θ is essentially constant throughout the motion.) What is the radius of the circle? What is the frequency of the motion?

Let $\hat{x}$ point east, $\hat{y}$ point north, $\hat{z}$ point "up" so that $\vec{g} = -g\hat{z}$.

Earth's rotation vector is $\vec{\Omega}$ where $|\Omega| \simeq \frac{2\pi}{86400\,s} \simeq 7.3 \times 10^{-5}\,s^{-1}$, and

$$\hat{\Omega} = \cos\theta\,\hat{z} + \sin\theta\,\hat{y}$$

$$\vec{F}_{coriolis} = 2m\vec{v} \times \vec{\Omega}$$

$$= 2m\left[(v_y\Omega_z - \underset{0}{v_z}\Omega_y)\hat{x} + (\underset{0}{v_z}\Omega_x - v_x\Omega_z)\hat{y} + (v_x\Omega_y - v_y\underset{0}{\Omega_x})\hat{z}\right]$$

Ignore F_z, since we're on frictionless ice, and since we'll assume $|v_x\Omega_y| \ll g$, so puck will not go airborne.

$$m\ddot{x} = 2m v_y \Omega_z = 2m\Omega\cos\theta\, v_y = m\dot{v}_x$$
$$m\ddot{y} = -2m v_x \Omega_z = -2m\Omega\cos\theta\, v_x = m\dot{v}_y$$

$$\dot{v}_x = (2\Omega\cos\theta)\, v_y \quad , \quad \dot{v}_y = -(2\Omega)\cos\theta\, v_x$$

let $\eta = v_x + i v_y = A e^{i\omega t} \longrightarrow \dot{\eta} = i\omega\eta$

$$\dot{v}_x + i\dot{v}_y = i\omega v_x - \omega v_y \longrightarrow \dot{v}_x = -\omega v_y , \; \dot{v}_y = \omega v_x$$
$$\Longrightarrow \boxed{\omega = -2\Omega\cos\theta}$$

$$x + iy = \frac{A}{i\omega}(\cos\omega t + i\sin\omega t) = \frac{|A|e^{i\delta}}{i\omega}(\cos\omega t + i\sin\omega t)$$

$$x = \frac{|A|}{\omega}\sin(\omega t + \delta) \qquad y = -\frac{|A|}{\omega}\cos(\omega t + \delta)$$

which describes a circle of radius $\frac{|A|}{|\omega|} = \boxed{\frac{v}{2\Omega\cos\theta}}$
and angular frequency $2\Omega\cos\theta$

If $\theta = 45°$, $v = 1\frac{m}{s}$, then $R \simeq 10\,km$! (small effect!)

$$m\boldsymbol{a} \; = \; \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

$$m\boldsymbol{a} \; = \; \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m\,\Omega^2 \rho\,\hat{\boldsymbol{\rho}}$$

Let's go through two more examples to try to gain more insight into the less intuitive Coriolis term.

First, a quick question: for an object in the northern hemisphere moving due north, the Coriolis force points due ... ?

$$m\boldsymbol{a} \;=\; \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$$
$$m\boldsymbol{a} \;=\; \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m\,\Omega^2\rho\,\hat{\boldsymbol{\rho}}$$

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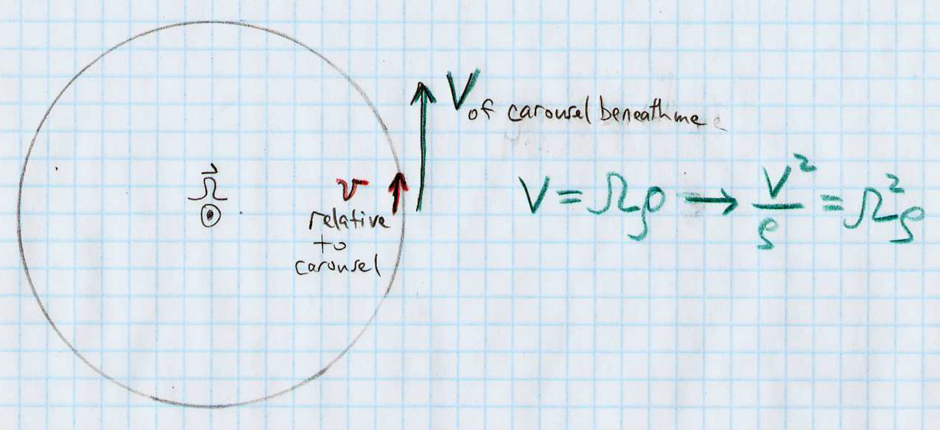
$$\hat{\boldsymbol{e}}_{\rm north} \;=\; \hat{\boldsymbol{\Omega}}\sin\theta - \hat{\boldsymbol{\rho}}\cos\theta$$

$$\hat{\boldsymbol{\Omega}}\times\hat{\boldsymbol{\Omega}} \;=\; 0 \qquad\qquad (-\hat{\boldsymbol{\rho}})\times\hat{\boldsymbol{\Omega}} \;=\; \hat{\boldsymbol{e}}_{\rm east}$$

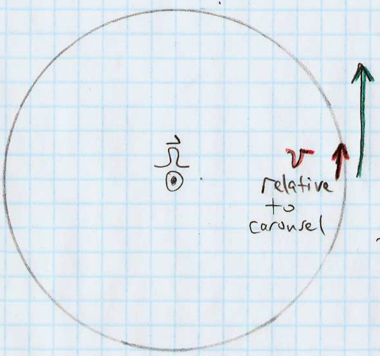
$$\hat{\boldsymbol{e}}_{\rm north}\times\hat{\boldsymbol{\Omega}} \;=\; (\cos\theta)\;\hat{\boldsymbol{e}}_{\rm east} \;=\; (\cos\theta)\;\hat{\boldsymbol{\phi}}$$

Centrifugal force only cares about ρ and always points in the $\hat{\boldsymbol{\rho}}$ direction. Coriolis force looks at $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ components of $\boldsymbol{v}$ and points the plane perpendicular to $\hat{\boldsymbol{\Omega}}$. So let's work in the $\perp$ plane, e.g. on a carousel.



If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet?

Evaluate this in the inertial frame, in terms of v, V, and ρ.



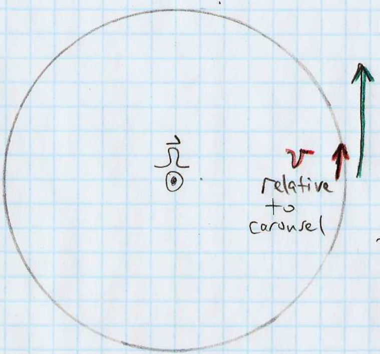
$$V = \Omega\rho \rightarrow \frac{V^2}{\rho} = \Omega^2\rho$$

$$\frac{mv_0^2}{\rho} = \frac{m(v+V)^2}{\rho} = \frac{mv^2}{\rho} + \frac{2mvV}{\rho} + \frac{mV^2}{\rho}$$

$$= \frac{mv^2}{\rho} + 2mv\Omega + m\Omega^2\rho$$

If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet? ("v_0" is w.r.t. inertial frame.)

Now interpret each of the three terms above!



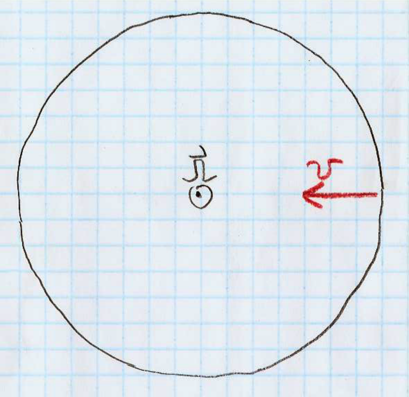
$$V = \Omega\rho \rightarrow \frac{V^2}{\rho} = \Omega^2\rho$$

$$\frac{m v_0^2}{\rho} = \frac{m(v+V)^2}{\rho} = \frac{mv^2}{\rho} + \frac{2mvV}{\rho} + \frac{mV^2}{\rho}$$

$$= \frac{mv^2}{\rho} + 2mv\Omega + m\Omega^2\rho$$

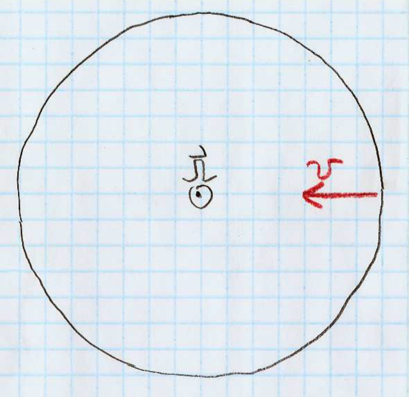
If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet? ("v_0" is w.r.t. inertial frame.)

How would the result change if I were instead walking tangentially at (relative) speed v opposite the carousel's direction of rotation?



Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Before I start to walk, when I'm just standing "still" at the outer radius of the carousel, what is the magnitude of the frictional force between the carousel floor and my feet? How does the balance of forces look in the rotating frame?



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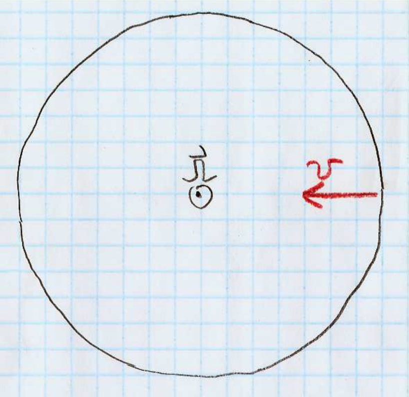
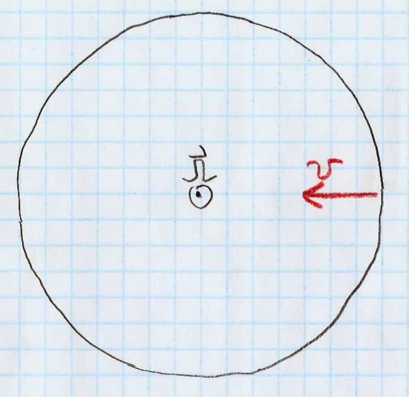
From perspective of carousel's rotating frame,

$$m\boldsymbol{a} = \boldsymbol{F}_{\text{friction}} + \boldsymbol{F}_{\text{centrifugal}} = 0$$



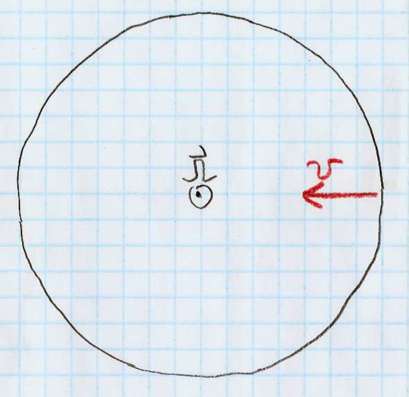
Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Let my mass be m and let my radial position be ρ. Before I start to walk, what is my angular momentum (which you should evaluate in the inertial frame)?



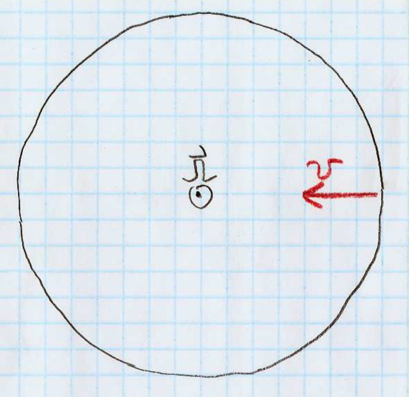
Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Let my mass be m and let my radial position be ρ. As I walk radially inward with constant speed v (in carousel frame), what is the rate of change of my angular momentum (which you should evaluate in the inertial frame)?



Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

What force (acting on my feet!) provides the torque that must equal the rate of change of my angular momentum as I walk inward? In what direction does that force point?



Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

In the rotating frame of the carousel, my tangential acceleration is zero. As I look from the frame of the carousel, what pseudoforce balances the tangential frictional force such that the net tangential acceleration is zero?

positron.hep.upenn.edu/p351/files/0223_morin_coriolis.pdf

I like the Coriolis force, because it's the least-familiar topic from Ch9. And I like the many places in Taylor's book where he uses "perturbation theory," since this is such a widely useful technique in physics.

So let's finish the problem we started Monday, which combines both ideas.

At a polar angle θ, a projectile is fired eastward with speed v_0 at an angle α above the ground. Let $\hat{x}$ point east, let $\hat{y}$ point north, and let $\hat{z}$ point "up." To first order in Ω, work out the effects, Δx and Δy, of the Coriolis force on the projectile's landing point. Ignore air resistance and ignore Earth's curvature.

$$m\boldsymbol{a} = \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m\,\Omega^2 \rho\,\hat{\boldsymbol{\rho}}$$

0^{th} order in Ω:

$$x = (v_0 \cos\alpha)t \qquad y = 0 \qquad z = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

$$z_f = 0 \Rightarrow t_f = \frac{2v_0 \sin\alpha}{g} \Rightarrow x_f = \frac{2v_0^2 \sin\alpha\cos\alpha}{g}$$

1^{st} order in Ω:

$$\hat{\Omega} = \hat{z}\cos\theta + \hat{y}\sin\theta$$

$$\underset{\sim}{F}_{\text{coriolis}} = 2m\underset{\sim}{v} \times \underset{\sim}{\Omega}$$

$$m\ddot{x} = 2m(\cancel{v_y}\Omega_z - v_z\Omega_y) = -2m(\Omega\sin\theta)(v_0\sin\alpha - gt)$$

$$m\ddot{y} = 2m(v_z\cancel{\Omega_x} - v_x\Omega_z) = -2m(\Omega\cos\theta)(v_0\cos\alpha)$$

$$m\ddot{z} = -mg + 2m(v_x\Omega_y - \cancel{v_y}\cancel{\Omega_x}) = -mg + 2m(\Omega\sin\theta)(v_0\cos\alpha)$$

(Continue from here.)

$$\ddot{z} = -g + 2V_0\Omega\sin\theta\cos\alpha$$
$$V_z = V_0\sin\alpha + (2V_0\Omega\sin\theta\cos\alpha - g)\,t$$
$$z = (V_0\sin\alpha)\,t + \tfrac{1}{2}(2V_0\Omega\sin\theta\cos\alpha - g)\,t^2$$

$$z_f = 0 \Rightarrow t_f = \frac{2V_0\sin\alpha}{g - 2V_0\Omega\sin\theta\cos\alpha} \simeq \frac{2V_0\sin\alpha}{g} + \frac{4V_0^2\Omega\sin\theta\sin\alpha\cos\alpha}{g^2}$$

$$\ddot{y} = -2V_0\Omega\cos\theta\cos\alpha$$
$$V_y = (-2V_0\Omega\cos\theta\cos\alpha)\,t$$
$$y = \tfrac{1}{2}(-2V_0\Omega\cos\theta\cos\alpha)\,t^2 = -(V_0\Omega\cos\theta\cos\alpha)\,t^2$$
$$y_f^{(\text{1st order})} = -(V_0\Omega\cos\theta\cos\alpha)\left(\frac{2V_0\sin\alpha}{g}\right)^2$$
$$\Delta y \equiv y_f^{(1)} - y_f^{(0)} = -\frac{4V_0^3\Omega\cos\theta}{g^2}\cos\alpha\sin^2\alpha$$

$$t_f = \frac{2v_0 \sin\alpha}{g - 2v_0 \Omega \sin\theta \cos\alpha} \simeq \left[t_f\right]_{\Omega=0} + \Omega \left[\frac{dt_f}{d\Omega}\right]_{\Omega=0}$$

$$\left[\frac{dt_f}{d\Omega}\right]_{\Omega=0} = \left[\frac{-2v_0 \sin\alpha\,(-2v_0 \sin\theta \cos\alpha)}{(g - 2v_0 \Omega \sin\theta \cos\alpha)^2}\right]_{\Omega=0} = \frac{4v_0^2 \sin\theta \sin\alpha \cos\alpha}{g^2}$$

$$t_f = \frac{2v_0 \sin\alpha}{g} + \frac{4v_0^2 \Omega \sin\theta \sin\alpha \cos\alpha}{g^2} + \mathcal{O}(\Omega^2)$$

$$\ddot{x} = -2v_0\Omega\sin\theta\sin\alpha + 2(\Omega\sin\theta)gt$$

$$v_x = v_0\cos\alpha - 2(v_0\Omega\sin\theta\sin\alpha)t + (\Omega\sin\theta)gt^2$$

$$x = (v_0\cos\alpha)t - (v_0\Omega\sin\theta\sin\alpha)t^2 + \frac{1}{3}(\Omega\sin\theta)gt^3$$

$$x_f^{(1\text{st order})} = (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha}{g} + \frac{4v_0^2\Omega\sin\theta\sin\alpha\cos\alpha}{g^2}\right)$$

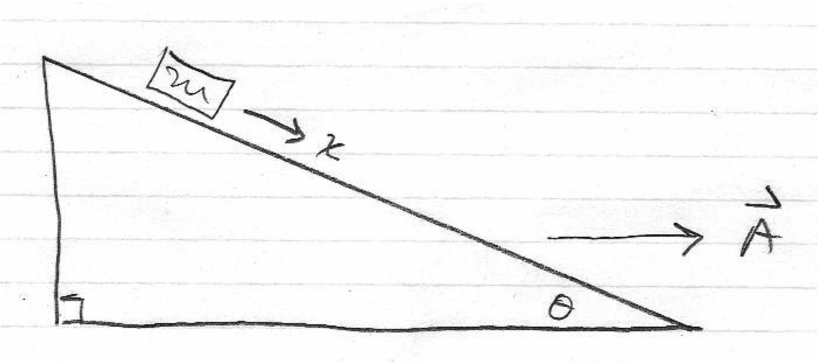
$$-(v_0\Omega\sin\theta\sin\alpha)\left(\frac{2v_0\sin\alpha}{g}\right)^2$$

$$+\frac{1}{3}(\Omega\sin\theta)g\left(\frac{2v_0\sin\alpha}{g}\right)^3$$

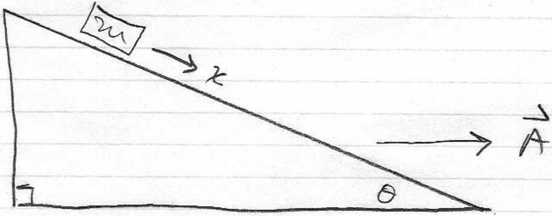
$$-\frac{12}{3}+\frac{8}{3} = -\frac{4}{3}$$

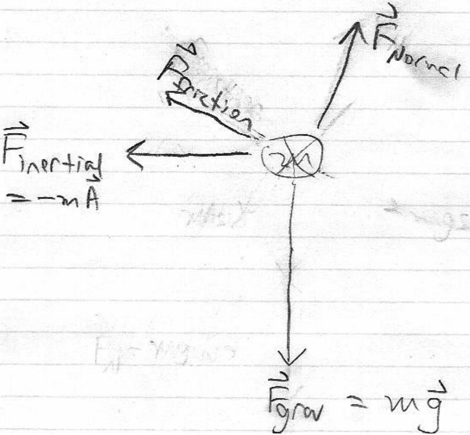
$$= \frac{2v_0^2\sin\alpha\cos\alpha}{g} + \frac{4v_0^3\Omega}{g^2}\sin\theta\sin\alpha\cos^2\alpha + \frac{v_0^3\Omega}{g^2}\sin\theta\sin^3\alpha\underbrace{\left(-4+\frac{8}{3}\right)}$$

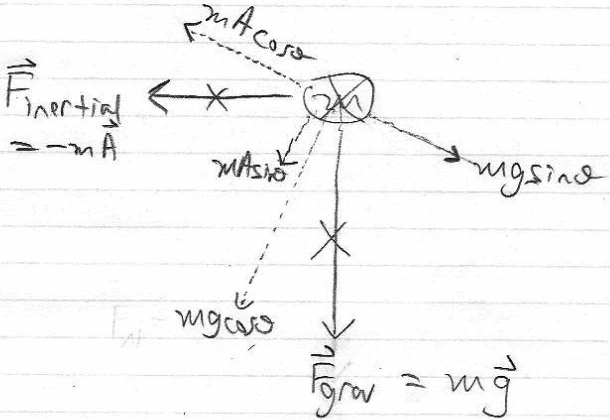
$$\Delta x \equiv x_f^{(1)} - x_f^{(0)} = \frac{4v_0^3\Omega}{g^2}\sin\theta\left(\sin\alpha\cos^2\alpha - \frac{1}{3}\sin^3\alpha\right)$$



What is smallest value of static friction coefficient μ for which the block can stand still on the ramp? (What's the familiar answer if $A = 0$?)


m
x
$\vec{A}$
θ


$\vec{F}_{Normal}$
$\vec{F}_{friction}$
$\vec{F}_{inertial} = -m\vec{A}$
m
$\vec{F}_{grav} = m\vec{g}$



$$F_N = mg\cos\theta + mA\sin\theta$$

$$F_F \leq \mu F_N = \mu m(g\cos\theta + A\sin\theta)$$

$$m\ddot{x} = m(g\sin\theta - A\cos\theta) - F_F$$

$$F_N = mg\cos\theta + mA\sin\theta$$

$$F_F \leq \mu F_N = \mu m(g\cos\theta + A\sin\theta)$$

$$m\ddot{x} = m(g\sin\theta - A\cos\theta) - F_F$$

block just starts to slip on ramp if

$$g\sin\theta - A\cos\theta = \mu(g\cos\theta + A\sin\theta)$$

$$\mu = \frac{g\sin\theta - A\cos\theta}{g\cos\theta + A\sin\theta} = \frac{\cos\beta\sin\theta - \sin\beta\cos\theta}{\cos\beta\cos\theta + \sin\beta\sin\theta}$$

$= \tan(\theta - \beta)$ where $\tan\beta = A/g$

Note that

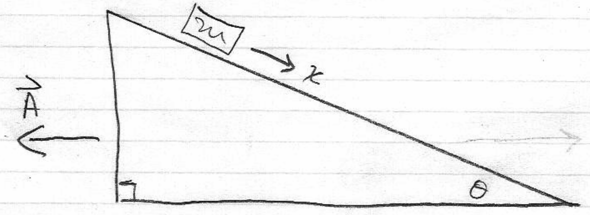
$$\tan(\theta - \beta) = \frac{\cos\beta\sin\theta - \sin\beta\cos\theta}{\cos\beta\cos\theta + \sin\beta\sin\theta}$$

Note that

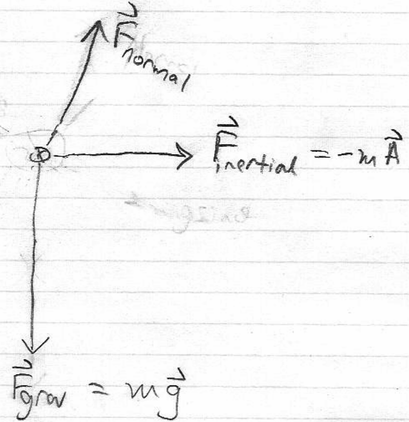
$$\tan(\theta - \beta) = \frac{\cos\beta\sin\theta - \sin\beta\cos\theta}{\cos\beta\cos\theta + \sin\beta\sin\theta}$$

```
In[6]:= Factor[TrigExpand[Tan[a-b]]]

        Cos[b] Sin[a] - Cos[a] Sin[b]
Out[6]= -----------------------------
        Cos[a] Cos[b] + Sin[a] Sin[b]
```



m
x
$\vec{A}$
θ



$\vec{F}_{normal}$
$\vec{F}_{inertial} = -m\vec{A}$
$\vec{F}_{grav} = m\vec{g}$

No friction, but wedge is recoiling with recoil acceleration $\vec{A}$ (to the left)

$$m\ddot{x} = m\left(g\sin\theta + A\cos\theta\right)$$

linear momentum conservation:

$$m\left(\ddot{x}\cos\theta - A\right) = MA \longrightarrow A = \frac{m\ddot{x}\cos\theta}{M+m}$$

$$m\ddot{x} = mg\sin\theta + \frac{m^2}{M+m}\ddot{x}\cos^2\theta$$

$$\ddot{x}\left(1 - \frac{m}{m+M}\cos^2\theta\right) = g\sin\theta$$

$$\ddot{x} = \frac{g\sin\theta}{1 - \frac{m}{m+M}\cos^2\theta}$$

Physics 351 — Friday, February 23, 2018

- You'll read the first half (§10.1–10.7) of Chapter 10 this weekend. We'll start talking about Ch10 mid/late next week.
- The midterm exam (**March** 26) will cover (only!) chapters 7,8,9. But we will be working on Ch 10 by then.
- Turn in HW5. Pick up handout for HW6, due next Friday.