Physics 351 — Wednesday, February 28, 2018

- HW6 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.

Let $\hat{x}$ point east, $\hat{y}$ point north, and $\hat{z}$ point up, for a point on Earth's surface at colatitude θ. Decompose Earth's rotation vector $\boldsymbol{\Omega}$ into its $\hat{y}$ and $\hat{z}$ components. Then write down the expresson for $\boldsymbol{F}_{\text{coriolis}}$ and write out, separately, the x, y, z components of $\boldsymbol{F}_{\text{coriolis}}$ in terms of v_x, v_y, v_z, Ω, $\cos\theta$, and $\sin\theta$.

Once you've done that, we'll return to Monday's problem of the Coriolis deflection (w.r.t. its order(Ω^0) landing spot) of a projectile fired eastward.

Here's a slightly trickier Coriolis projectile problem. What makes this problem tricky is that the Coriolis force also has a first-order effect on the projectile's flight time.

At a polar angle θ, a projectile is fired eastward with speed v_0 at an angle α above the ground. Let $\hat{x}$ point east, let $\hat{y}$ point north, and let $\hat{z}$ point "up." To first order in Ω, work out the effects, Δx and Δy, of the Coriolis force on the projectile's landing point. Ignore air resistance and ignore Earth's curvature.

$$m\boldsymbol{a} = \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m\,\Omega^2 \rho\,\hat{\boldsymbol{\rho}}$$

0^{th} order in Ω:

$$x = (v_0 \cos\alpha)t \qquad y = 0 \qquad z = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

$$z_f = 0 \Rightarrow t_f = \frac{2v_0 \sin\alpha}{g} \Rightarrow x_f = \frac{2v_0^2 \sin\alpha\cos\alpha}{g}$$

1^{st} order in Ω:

$$\hat{\Omega} = \hat{z}\cos\theta + \hat{y}\sin\theta$$

$$\underset{\sim}{F}_{coriolis} = 2m\underset{\sim}{v} \times \underset{\sim}{\Omega}$$

$$m\ddot{x} = 2m(\cancel{v_y \Omega_z} - v_z\Omega_y) = -2m(\Omega\sin\theta)(v_0\sin\alpha - gt)$$

$$m\ddot{y} = 2m(\cancel{v_z\Omega_x} - v_x\Omega_z) = -2m(\Omega\cos\theta)(v_0\cos\alpha)$$

$$m\ddot{z} = -mg + 2m(v_x\Omega_y - \cancel{v_y\Omega_x}) = -mg + 2m(\Omega\sin\theta)(v_0\cos\alpha)$$

$$\ddot{z} = -g + 2v_0\Omega\sin\theta\cos\alpha$$
$$v_z = v_0\sin\alpha + (2v_0\Omega\sin\theta\cos\alpha - g)\,t$$
$$z = (v_0\sin\alpha)\,t + \tfrac{1}{2}(2v_0\Omega\sin\theta\cos\alpha - g)\,t^2$$

$$z_f = 0 \Rightarrow t_f = \frac{2v_0\sin\alpha}{g - 2v_0\Omega\sin\theta\cos\alpha} \simeq \frac{2v_0\sin\alpha}{g} + \frac{4v_0^2\Omega\sin\theta\sin\alpha\cos\alpha}{g^2}$$

$$\ddot{y} = -2v_0\Omega\cos\theta\cos\alpha$$
$$v_y = (-2v_0\Omega\cos\theta\cos\alpha)\,t$$
$$y = \tfrac{1}{2}(-2v_0\Omega\cos\theta\cos\alpha)\,t^2 = -(v_0\Omega\cos\theta\cos\alpha)\,t^2$$
$$y_f^{(\text{1st order})} = -(v_0\Omega\cos\theta\cos\alpha)\left(\frac{2v_0\sin\alpha}{g}\right)^2$$
$$\Delta y \equiv y_f^{(1)} - y_f^{(0)} = -\frac{4v_0^3\Omega\cos\theta}{g^2}\cos\alpha\sin^2\alpha$$

$$t_f = \frac{2v_0 \sin\alpha}{g - 2v_0 \Omega \sin\theta \cos\alpha} \simeq \left[t_f\right]_{\Omega=0} + \Omega \left[\frac{dt_f}{d\Omega}\right]_{\Omega=0}$$

$$\left[\frac{dt_f}{d\Omega}\right]_{\Omega=0} = \left[\frac{-2v_0 \sin\alpha\,(-2v_0 \sin\theta \cos\alpha)}{(g - 2v_0 \Omega \sin\theta \cos\alpha)^2}\right]_{\Omega=0} = \frac{4v_0^2 \sin\theta \sin\alpha \cos\alpha}{g^2}$$

$$t_f = \frac{2v_0 \sin\alpha}{g} + \frac{4v_0^2 \Omega \sin\theta \sin\alpha \cos\alpha}{g^2} + \mathcal{O}(\Omega^2)$$

$$\ddot{x} = -2v_0\Omega\sin\theta\sin\alpha + 2(\Omega\sin\theta)gt$$

$$v_x = v_0\cos\alpha - 2(v_0\Omega\sin\theta\sin\alpha)t + (\Omega\sin\theta)gt^2$$

$$x = (v_0\cos\alpha)t - (v_0\Omega\sin\theta\sin\alpha)t^2 + \frac{1}{3}(\Omega\sin\theta)gt^3$$

$$\begin{aligned}
x_f^{(1st\ order)} &= (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha}{g} + \frac{4v_0^2\Omega\sin\theta\sin\alpha\cos\alpha}{g^2}\right) \\
&\quad - (v_0\Omega\sin\theta\sin\alpha)\left(\frac{2v_0\sin\alpha}{g}\right)^2 \\
&\quad + \frac{1}{3}(\Omega\sin\theta)g\left(\frac{2v_0\sin\alpha}{g}\right)^3
\end{aligned}$$

$$-\frac{12}{3} + \frac{8}{3} = -\frac{4}{3}$$

$$= \frac{2v_0^2\sin\alpha\cos\alpha}{g} + \frac{4v_0^3\Omega}{g^2}\sin\theta\sin\alpha\cos^2\alpha + \frac{v_0^3\Omega}{g^2}\sin\theta\sin^3\alpha\underbrace{\left(-4 + \frac{8}{3}\right)}$$

$$\Delta x \equiv x_f^{(1)} - x_f^{(0)} = \frac{4v_0^3\Omega}{g^2}\sin\theta\left(\sin\alpha\cos^2\alpha - \frac{1}{3}\sin^3\alpha\right)$$

$$m\boldsymbol{a} = \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

$$m\boldsymbol{a} = \boldsymbol{F} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} + m\,\Omega^2 \rho\,\hat{\boldsymbol{\rho}}$$

Let's go through two more examples to try to gain more insight into the less intuitive Coriolis term.

First, a quick question: for an object in the northern hemisphere moving due north, the Coriolis force points due ... ?

$$m\boldsymbol{a} \;=\; \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$$

$$m\boldsymbol{a} \;=\; \boldsymbol{F} + 2m\boldsymbol{v}\times\boldsymbol{\Omega} + m\,\Omega^2\rho\,\hat{\boldsymbol{\rho}}$$

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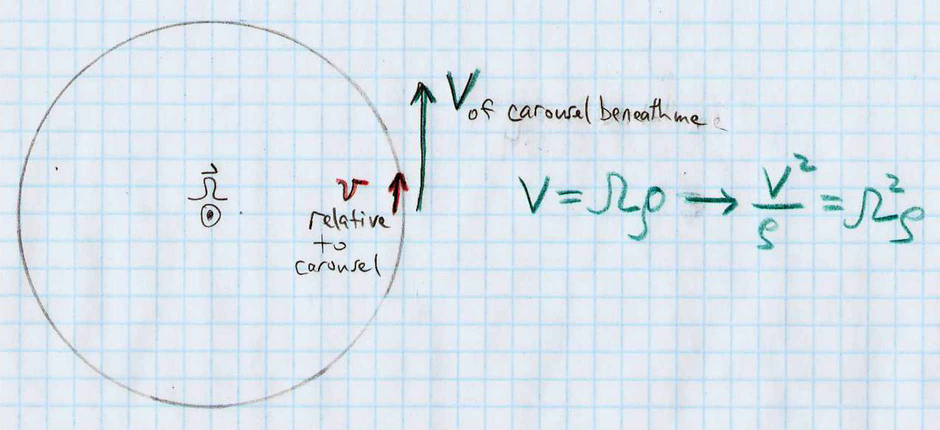
$$\hat{\boldsymbol{e}}_{\rm north} \;=\; \hat{\boldsymbol{\Omega}}\sin\theta - \hat{\boldsymbol{\rho}}\cos\theta$$

$$\hat{\boldsymbol{\Omega}}\times\hat{\boldsymbol{\Omega}} \;=\; 0 \qquad\qquad (-\hat{\boldsymbol{\rho}})\times\hat{\boldsymbol{\Omega}} \;=\; \hat{\boldsymbol{e}}_{\rm east}$$

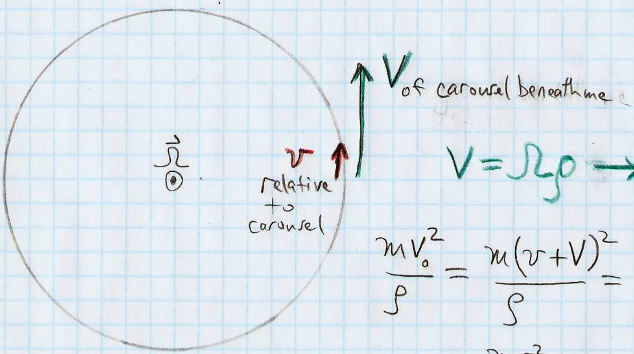
$$\hat{\boldsymbol{e}}_{\rm north}\times\hat{\boldsymbol{\Omega}} \;=\; (\cos\theta)\;\hat{\boldsymbol{e}}_{\rm east} \;=\; (\cos\theta)\;\hat{\boldsymbol{\phi}}$$

Centrifugal force only cares about ρ and always points in the $\hat{\boldsymbol{\rho}}$ direction. Coriolis force looks at $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ components of $\boldsymbol{v}$ and it lies in the plane perpendicular to $\hat{\boldsymbol{\Omega}}$. So let's work in the $\perp$ plane, e.g. on a carousel.



If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet?

Evaluate this in the inertial frame, in terms of v, V, and ρ.



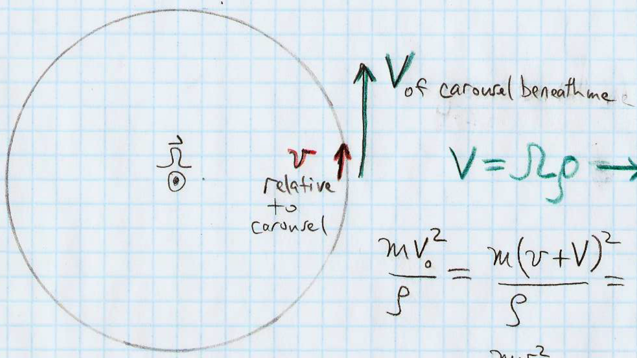
$$V = \Omega\rho \rightarrow \frac{V^2}{\rho} = \Omega^2\rho$$

$$\frac{m v_0^2}{\rho} = \frac{m(v+V)^2}{\rho} = \frac{mv^2}{\rho} + \frac{2mvV}{\rho} + \frac{mV^2}{\rho}$$

$$= \frac{mv^2}{\rho} + 2mv\Omega + m\Omega^2\rho$$

If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet? ("v_0" is w.r.t. inertial frame.)

Now interpret each of the three terms above, from the perspective of the rotating frame.



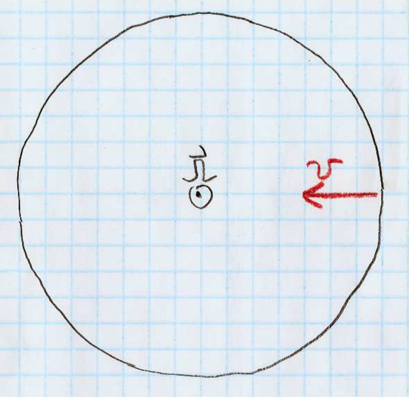
$$V=\Omega\rho \rightarrow \frac{V^2}{\rho}=\Omega^2\rho$$

$$\frac{mv_0^2}{\rho}=\frac{m(v+V)^2}{\rho}=\frac{mv^2}{\rho}+\frac{2mvV}{\rho}+\frac{mV^2}{\rho}$$

$$=\frac{mv^2}{\rho}+2mv\Omega+m\Omega^2\rho$$

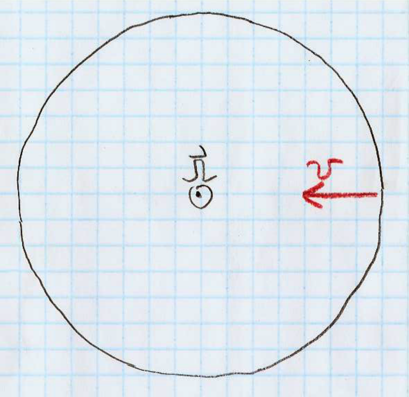
If I am standing on the carousel and I want to move tangentially at constant speed v w.r.t. the rotating frame of the carousel (so I'm at constant radius ρ), how big must be the radial force of friction between the carousel and my feet? ("v_0" is w.r.t. inertial frame.)

How would the result change if I were instead walking tangentially at (relative) speed v opposite the carousel's direction of rotation?



Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Before I start to walk, when I'm just standing "still" at the outer radius of the carousel, what is the magnitude of the frictional force between the carousel floor and my feet? How does the balance of forces look in the rotating frame?



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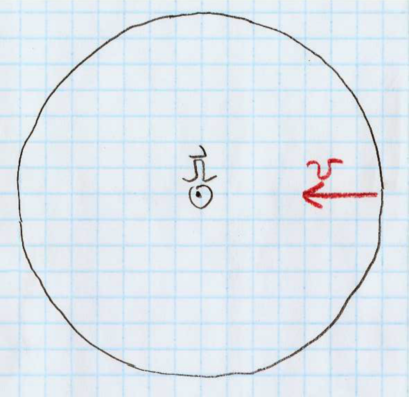
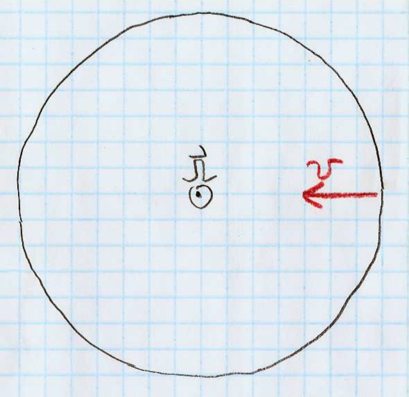
From perspective of carousel's rotating frame,

$$m\boldsymbol{a} = \boldsymbol{F}_{\text{friction}} + \boldsymbol{F}_{\text{centrifugal}} = 0$$



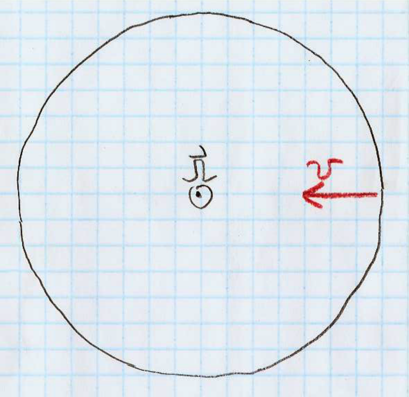
Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Let my mass be m and let my radial position be ρ. Before I start to walk, what is my angular momentum (which you should evaluate in the inertial frame)?



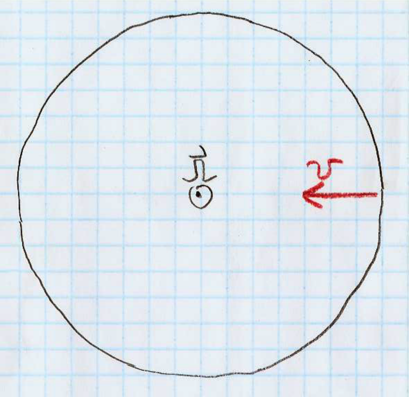
Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

Let my mass be m and let my radial position be ρ. As I walk radially inward with constant speed v (in carousel frame), what is the rate of change of my angular momentum (which you should evaluate in the inertial frame)?



Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

What [real] force (acting on my feet!) provides the torque that must equal the rate of change of my angular momentum as I walk inward? In what direction does that force point?



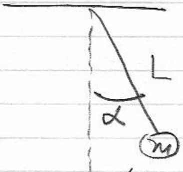
Now suppose I want to walk radially inward in the frame of the carousel, i.e. I want to walk along a radial line painted on the floor of the carousel.

In the rotating frame of the carousel, my tangential acceleration is zero. As I look from the frame of the carousel, what pseudoforce balances the tangential frictional force such that the net tangential acceleration is zero?

http://positron.hep.upenn.edu/p351/files/0223_morin_coriolis.pdf

Let's try a Lagrangian version of the Foucault pendulum.

$$\boldsymbol{\Omega} = \hat{z}\cos\theta + \hat{y}\sin\theta$$



$$\boldsymbol{\Omega} \times \hat{x} = \Omega\cos\theta\,\hat{y} - \Omega\sin\theta\,\hat{z}$$

$$\boldsymbol{\Omega} \times \hat{y} = -\Omega\cos\theta\,\hat{x}$$

$$\boldsymbol{\Omega} \times \hat{z} = \Omega\sin\theta\,\hat{x}$$

$$U = mgL(1-\cos\alpha)$$
$$\simeq mgL\frac{\alpha^2}{2}$$

$$\boldsymbol{r} = x\hat{x} + y\hat{y} + R\hat{z}$$

$$\dot{\boldsymbol{r}}_0 = \dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r} = \dot{x}\hat{x} + \dot{y}\hat{y} + \Omega x(\hat{y}\cos\theta - \hat{z}\sin\theta)$$
$$- \Omega\cos\theta\, y\,\hat{x} + \Omega\sin\theta\, R\hat{x}$$

$$\dot{\boldsymbol{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

$$|\dot{\boldsymbol{r}}_0|^2 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)^2 + (\dot{y} + x\Omega\cos\theta)^2 + (x\Omega\sin\theta)^2$$

(drop any Ω^2 terms)

$$\dot{\underset{\sim}{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

$$|\dot{\underset{\sim}{r}}_0|^2 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)^2 + (\dot{y} + x\Omega\cos\theta)^2 + (x\Omega\sin\theta)^2$$

(drop any Ω^2 terms)

$$|\dot{\underset{\sim}{r}}_0|^2 = \dot{x}^2 - 2\dot{x}y\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta + \dot{y}^2 + 2\dot{y}x\Omega\cos\theta$$

$$= \dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta$$

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right]$$

$$U \simeq mg\frac{(\alpha L)^2}{2L} = mg\frac{(x^2+y^2)}{2L} = \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}(x^2+y^2)$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) \Rightarrow m\dot{y}\Omega\cos\theta - \frac{mgx}{L} = \frac{d}{dt}\left(m\dot{x} - my\Omega\cos\theta + 2R\Omega\sin\theta\right)$$

$$\ddot{x} - \dot{y}\Omega\cos\theta = \dot{y}\Omega\cos\theta - \frac{gx}{L} \Rightarrow \ddot{x} = -\frac{g}{L}x + 2\Omega\cos\theta\,\dot{y}$$

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) \Rightarrow -m\dot{x}\Omega\cos\theta - \frac{mgy}{L} = \frac{d}{dt}\left(m\dot{y} + mx\Omega\cos\theta\right) = m\ddot{y} + m\dot{x}\Omega\cos\theta$$

$$\left.\begin{array}{l} \ddot{y} = -\frac{g}{L}y - 2\Omega\cos\theta\,\dot{x} \\ \text{let } \Omega_y = \Omega\cos\theta \end{array}\right\} \begin{array}{l} \ddot{x} = -\frac{g}{L}x + 2\Omega_y\dot{y} \\ \ddot{y} = -\frac{g}{L}y - 2\Omega_y\dot{x} \end{array}$$

try $\eta \equiv x + iy = Ae^{iBt} \Rightarrow \dot{\eta} = iB\eta, \quad \ddot{\eta} = -B^2\eta$

$$\ddot{x} = -\omega_0^2 x + 2\Omega_y \dot{y} \qquad \ddot{y} = -\omega_0^2 y - 2\Omega_y \dot{x}$$

$$\ddot{x} + i\ddot{y} = -\omega_0^2(x+iy) + 2\Omega_y(\dot{y} - i\dot{x}) = -\omega_0^2(x+iy) - 2i\Omega_y(\dot{x} + i\dot{y})$$

$$\ddot{\eta} = -\omega_0^2\eta - 2i\Omega_y\dot{\eta} \Rightarrow -B^2 = -\omega_0^2 - (2i\Omega_y)(iB)$$

$$-B^2 = -\omega_0^2 + 2\Omega_y B \Rightarrow B^2 + 2B\Omega_y - \omega_0^2 = 0$$

$$B = \frac{1}{2}\left(-2\Omega_y \pm \sqrt{4\Omega_y^2 + 4\omega_0^2}\right) = -\Omega_y \pm \sqrt{\Omega_y^2 + \omega_0^2} \simeq -\Omega_y \pm \omega_0$$

$$x + iy = A_1 e^{-i\Omega_y t} e^{i\omega_0 t} + A_2 e^{-i\Omega_y t} e^{-i\omega_0 t}$$

$$= e^{-i\Omega_y t}\left(C_1 \cos\omega_0 t + \cancel{C_2 \sin\omega_0 t}\right)$$

$$x(t) = C\cos(\Omega_y t)\cos(\omega_0 t)$$

$$y(t) = -C\sin(\Omega_y t)\cos(\omega_0 t)$$

A puck slides with speed v on frictionless ice. The surface is "level" in the sense that it is orthogonal to the effective (gravitational + centrifugal) $\boldsymbol{g}$ at all points. Show that the puck moves in a circle, as seen in Earth's rotating frame. (Assume that v is small enough that the radius of the circle is much smaller than the radius of Earth, so that the colatitude θ is essentially constant throughout the motion.) What is the radius of the circle? What is the frequency of the motion?

Let $\hat{x}$ point east, $\hat{y}$ point north, $\hat{z}$ point "up" so that $\vec{g} = -g\hat{z}$.

Earth's rotation vector is $\vec{\Omega}$ where $|\Omega| \simeq \frac{2\pi}{86400s} \simeq 7.3 \times 10^{-5}\,s^{-1}$, and

$$\hat{\Omega} = \cos\theta\,\hat{z} + \sin\theta\,\hat{y}$$

$$\vec{F}_{coriolis} = 2m\vec{v} \times \vec{\Omega}$$

$$= 2m\left[(v_y\Omega_z - \underset{0}{v_z}\Omega_y)\hat{x} + (\underset{0}{v_z}\Omega_x - v_x\Omega_z)\hat{y} + (v_x\Omega_y - v_y\underset{0}{\Omega_x})\hat{z}\right]$$

Ignore F_z, since we're on frictionless ice, and since we'll assume $|v_x\Omega_y| \ll g$, so puck will not go airborne.

$$m\ddot{x} = 2m v_y \Omega_z = 2m\Omega\cos\theta\, v_y = m\dot{v}_x$$
$$m\ddot{y} = -2m v_x \Omega_z = -2m\Omega\cos\theta\, v_x = m\dot{v}_y$$

$$\dot{v}_x = (2\Omega\cos\theta)\, v_y \quad , \quad \dot{v}_y = -(2\Omega)\cos\theta\, v_x$$

let $\eta = v_x + i v_y = A e^{i\omega t} \longrightarrow \dot{\eta} = i\omega\eta$

$$\dot{v}_x + i\dot{v}_y = i\omega v_x - \omega v_y \longrightarrow \dot{v}_x = -\omega v_y \, , \; \dot{v}_y = \omega v_x$$
$$\Longrightarrow \boxed{\omega = -2\Omega\cos\theta}$$

$$x + iy = \frac{A}{i\omega}(\cos\omega t + i\sin\omega t) = \frac{|A| e^{i\delta}}{i\omega}(\cos\omega t + i\sin\omega t)$$

$$x = \frac{|A|}{\omega}\sin(\omega t + \delta) \qquad y = -\frac{|A|}{\omega}\cos(\omega t + \delta)$$

which describes a circle of radius $\frac{|A|}{|\omega|} = \boxed{\frac{v}{2\Omega\cos\theta}}$
and angular frequency $2\Omega\cos\theta$

If $\theta = 45°$, $v = 1\frac{m}{s}$, then $R \simeq 10\,\mathrm{km}$! (small effect!)

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