Physics 351 — Friday, March 2, 2018

- Turn in HW6. HW7 will be due the Friday after spring break (3/16), but I haven't written it up yet. I will post it online early next week.
- Read the rest of Ch10 (and answer online questions) during spring break, or as soon as you get back.

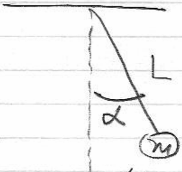
Let's try a Lagrangian version of the Foucault pendulum.

$$\underset{\sim}{\Omega} = \hat{z}\cos\theta + \hat{y}\sin\theta$$

$$\underset{\sim}{\Omega} \times \hat{x} = \Omega\cos\theta\,\hat{y} - \Omega\sin\theta\,\hat{z}$$

$$\underset{\sim}{\Omega} \times \hat{y} = -\Omega\cos\theta\,\hat{x}$$

$$\underset{\sim}{\Omega} \times \hat{z} = \Omega\sin\theta\,\hat{x}$$



$$U = mgL(1-\cos\alpha)$$
$$\simeq mgL\frac{\alpha^2}{2}$$

$$\underset{\sim}{r} = x\hat{x} + y\hat{y} + R\hat{z}$$

$$\dot{\underset{\sim}{r}}_0 = \dot{\underset{\sim}{r}} + \underset{\sim}{\Omega} \times \underset{\sim}{r} = \dot{x}\hat{x} + \dot{y}\hat{y} + \Omega \times (\hat{y}\cos\theta - \hat{z}\sin\theta)$$
$$- \Omega\cos\theta\, y\,\hat{x} + \Omega\sin\theta\, R\hat{x}$$

$$\dot{\underset{\sim}{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

$$|\dot{\underset{\sim}{r}}_0|^2 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)^2 + (\dot{y} + x\Omega\cos\theta)^2 + (x\Omega\sin\theta)^2$$

(drop any Ω^2 terms)

$$\dot{\underset{\sim}{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

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(drop any Ω^2 terms)

$$|\dot{\underset{\sim}{r}}_0|^2 = \dot{x}^2 - 2\dot{x}y\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta + \dot{y}^2 + 2\dot{y}x\Omega\cos\theta$$

$$= \dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta$$

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right]$$

$$U \simeq mg\frac{(\alpha L)^2}{2L} = mg\frac{(x^2+y^2)}{2L} = \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}\left(x^2+y^2\right)$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) \Rightarrow m\dot{y}\Omega\cos\theta - \frac{mgx}{L} = \frac{d}{dt}\left(m\dot{x} - my\Omega\cos\theta + 2R\Omega\sin\theta\right)$$

$$\ddot{x} - \dot{y}\Omega\cos\theta = \dot{y}\Omega\cos\theta - \frac{gx}{L} \Rightarrow \ddot{x} = -\frac{g}{L}x + 2\Omega\cos\theta\,\dot{y}$$

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) \Rightarrow -m\dot{x}\Omega\cos\theta - \frac{mgy}{L} = \frac{d}{dt}\left(m\dot{y} + mx\Omega\cos\theta\right) = m\ddot{y} + m\dot{x}\Omega\cos\theta$$

$$\ddot{y} = -\frac{g}{L}y - 2\Omega\cos\theta\,\dot{x}$$

let $\Omega_y = \Omega\cos\theta$

$$\left.\right\} \quad \begin{aligned} \ddot{x} &= -\frac{g}{L}x + 2\Omega_y\dot{y} \\ \ddot{y} &= -\frac{g}{L}y - 2\Omega_y\dot{x} \end{aligned}$$

try $\eta \equiv x + iy = Ae^{iBt} \Rightarrow \dot{\eta} = iB\eta\,,\quad \ddot{\eta} = -B^2\eta$

$$\ddot{x} = -\omega_0^2 x + 2\Omega_y \dot{y} \qquad \ddot{y} = -\omega_0^2 y - 2\Omega_y \dot{x}$$

$$\ddot{x} + i\ddot{y} = -\omega_0^2(x+iy) + 2\Omega_y(\dot{y} - i\dot{x}) = -\omega_0^2(x+iy) - 2i\Omega_y(\dot{x} + i\dot{y})$$

$$\ddot{\eta} = -\omega_0^2\eta - 2i\Omega_y\dot{\eta} \Rightarrow -B^2 = -\omega_0^2 - (2i\Omega_y)(iB)$$

$$-B^2 = -\omega_0^2 + 2\Omega_y B \Rightarrow B^2 + 2B\Omega_y - \omega_0^2 = 0$$

$$B = \frac{1}{2}\left(-2\Omega_y \pm \sqrt{4\Omega_y^2 + 4\omega_0^2}\right) = -\Omega_y \pm \sqrt{\Omega_y^2 + \omega_0^2} \simeq -\Omega_y \pm \omega_0$$

$$x + iy = A_1 e^{-i\Omega_y t} e^{i\omega_0 t} + A_2 e^{-i\Omega_y t} e^{-i\omega_0 t}$$

$$= e^{-i\Omega_y t}\left(C_1 \cos\omega_0 t + \cancel{C_2 \sin} \omega_0 t\right)$$

$$x(t) = C\cos(\Omega_y t)\cos(\omega_0 t)$$

$$y(t) = -C\sin(\Omega_y t)\cos(\omega_0 t)$$

A puck slides with speed v on frictionless ice. The surface is "level" in the sense that it is orthogonal to the effective (gravitational + centrifugal) $\boldsymbol{g}$ at all points. Show that the puck moves in a circle, as seen in Earth's rotating frame. (Assume that v is small enough that the radius of the circle is much smaller than the radius of Earth, so that the colatitude θ is essentially constant throughout the motion.) What is the radius of the circle? What is the frequency of the motion?

Let $\hat{x}$ point east, $\hat{y}$ point north, $\hat{z}$ point "up" so that $\vec{g} = -g\hat{z}$.

Earth's rotation vector is $\vec{\Omega}$ where $|\Omega| \simeq \frac{2\pi}{86400s} \simeq 7.3 \times 10^{-5}\,s^{-1}$, and

$$\hat{\Omega} = \cos\theta\,\hat{z} + \sin\theta\,\hat{y}$$

$$\vec{F}_{coriolis} = 2m\vec{v} \times \vec{\Omega}$$

$$= 2m\left[(v_y\Omega_z - \underset{0}{v_z}\Omega_y)\hat{x} + (\underset{0}{v_z}\Omega_x - v_x\Omega_z)\hat{y} + (v_x\Omega_y - v_y\underset{0}{\Omega_x})\hat{z}\right]$$

Ignore F_z, since we're on frictionless ice, and since we'll assume $|v_x\Omega_y| \ll g$, so puck will not go airborne.

$$m\ddot{x} = 2m v_y \Omega_z = 2m\Omega\cos\theta\, v_y = m\dot{v}_x$$
$$m\ddot{y} = -2m v_x \Omega_z = -2m\Omega\cos\theta\, v_x = m\dot{v}_y$$

$$\dot{v}_x = (2\Omega\cos\theta)\, v_y \quad , \quad \dot{v}_y = -(2\Omega)\cos\theta\, v_x$$

let $\eta = v_x + i v_y = A e^{i\omega t} \longrightarrow \dot{\eta} = i\omega\eta$

$$\dot{v}_x + i\dot{v}_y = i\omega v_x - \omega v_y \longrightarrow \dot{v}_x = -\omega v_y \, , \; \dot{v}_y = \omega v_x$$
$$\Longrightarrow \boxed{\omega = -2\Omega\cos\theta}$$

$$x + iy = \frac{A}{i\omega}(\cos\omega t + i\sin\omega t) = \frac{|A|e^{i\delta}}{i\omega}(\cos\omega t + i\sin\omega t)$$

$$x = \frac{|A|}{\omega}\sin(\omega t+\delta) \qquad y = -\frac{|A|}{\omega}\cos(\omega t+\delta)$$

which describes a circle of radius $\frac{|A|}{|\omega|} = \boxed{\frac{v}{2\Omega\cos\theta}}$
and angular frequency $2\Omega\cos\theta$

If $\theta = 45^\circ$, $v = 1\,\frac{m}{s}$, then $R \simeq 10\,\text{km}$! (small effect!)

I put this in the Jan 19 notes for reference, but decided it was too tedious to go through in class. In retrospect, I think it's a useful trick to know how to use, so let's do it today.

By the way, there is a fun (and at first glance slightly mysterious) way to prove the dreaded "BAC-CAB rule," using the "Cartesian Einstein notation."

Cartesian Einstein notation

vector $\vec{r} = (x, y, z) = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \sum_i r_i \hat{e}_i$

the i^{th} component $(i \in \{1,2,3\})$ of vector $\vec{A}$ is A_i

Kronecker delta : $\delta_{ij} = 1$ if $i = j$, else 0

dot product : $\vec{A} \cdot \vec{B} = \sum_{ij} A_i B_j \delta_{ij} = \sum_i A_i B_i$

matrix $\cdot$ vector : $\underline{\underline{M}} \cdot \vec{r} = \sum_{ij} M_{ij} r_j \hat{e}_i$

$$\left(\underline{\underline{M}} \cdot \vec{r}\right)_i = \sum_j M_{ij} r_j$$

matrix multiply : $\left(\underline{\underline{M}} \cdot \underline{\underline{N}}\right)_{ij} = \sum_k M_{ik} N_{kj}$

http://positron.hep.upenn.edu/p351/files/0119_cartesian_einstein.pdf

Levi-Civita symbol (a.k.a. permutation symbol, antisymmetric symbol)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \in \{123, 231, 312\} \\ -1 & \text{if } ijk \in \{213, 321, 132\} \\ 0 & \text{otherwise} \end{cases}$$

So $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$
all others are zero.

Cross product: $\vec{A} \times \vec{B} = \sum_{ijk} A_i B_j \hat{e}_k \varepsilon_{ijk}$

$$\vec{A} \times \vec{B} = (A_1 B_2 - A_2 B_1)\hat{e}_3 + (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$$

$$(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$$

Incredibly useful identity:

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Now use it to prove the dreaded "BAC-CAB" rule:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \sum_{ijk} A_i (\vec{B}\times\vec{C})_j \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklmn} A_i \left(B_l C_m \hat{e}_n \varepsilon_{lmn}\right)_j \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i \left(B_l C_m \varepsilon_{lmj}\right) \hat{e}_k \varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k \left(\varepsilon_{lmj}\varepsilon_{ijk}\right)$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k \left(\varepsilon_{lmj} \varepsilon_{ijk} \right)$$

$$\varepsilon_{lmj} \varepsilon_{ijk} = \varepsilon_{lmj} \varepsilon_{kij} = \left(\delta_{lk} \delta_{mi} - \delta_{li} \delta_{mk} \right)$$

$$= \sum_{ijklm} \left(A_i B_l C_m \hat{e}_k \delta_{lk} \delta_{mi} - A_i B_l C_m \hat{e}_k \delta_{li} \delta_{mk} \right)$$

$$= \sum_{ik} \left(A_i B_k C_i \hat{e}_k - A_i B_i C_k \hat{e}_k \right)$$

$$= \left(\sum_i A_i C_i \right) \left(\sum_k B_k \hat{e}_k \right) - \left(\sum_i A_i B_i \right) \left(\sum_k C_k \hat{e}_k \right)$$

$$= \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C}$$

$$= \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

Let's apply this technique to HW problem 10. To write the kinetic energy w.r.t. the inertial (non-rotating) frame, we use

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

which gives us

$$\boldsymbol{v}_o = \boldsymbol{v} + \Omega \times \boldsymbol{r} = \dot{\boldsymbol{r}} + \Omega \times \boldsymbol{r}$$

where $\boldsymbol{v}_o$ is the velocity in the inertial ("space") frame, while $\boldsymbol{v}$ is the velocity in the rotating ("body") frame. Now we can use $\boldsymbol{v}_o$ to write the KE w.r.t. the inertial frame:

$$\mathcal{L} = \frac{1}{2}m|\boldsymbol{v}_o|^2 - U(\boldsymbol{r}) = \frac{1}{2}m|\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r}|^2 - U(\boldsymbol{r})$$

Writing out the KE component by component (i=x,y,z):

$$\mathcal{L} = \left[\sum_i \frac{m}{2}(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i^2\right] - U(\boldsymbol{r})$$

Now pick coordinate n and differentiate $\mathcal{L}$. As usual, $\partial A^2/\partial r_n = (2A)(\partial A/\partial r_n)$.

$$\frac{\partial \mathcal{L}}{\partial r_n} = \left[\sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \frac{\partial}{\partial r_n}(\dot{r}_i + (\boldsymbol{\Omega} \times \boldsymbol{r})_i)\right] - \frac{\partial U(\boldsymbol{r})}{\partial r_n}$$

where I rewrote $(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i$ as $\dot{r}_i + (\boldsymbol{\Omega} \times \boldsymbol{r})_i$. The derivative of the first term is zero: $\partial \dot{r}_i/\partial r_n = 0$. We can write out the second term using the Cartesian Einstein notation as

$$(\boldsymbol{\Omega} \times \boldsymbol{r})_i = \sum_{jk} \Omega_j r_k \epsilon_{ijk}$$

whose derivative is

$$\frac{\partial}{\partial r_n}(\boldsymbol{\Omega} \times \boldsymbol{r})_i = \sum_{jk} \Omega_j \left(\frac{\partial r_k}{\partial r_n}\right) \epsilon_{ijk} = \sum_{jk} \Omega_j \delta_{kn} \epsilon_{ijk} = \sum_j \Omega_j \epsilon_{ijn}$$

Now we can plug this in to $\partial \mathcal{L}/\partial r_n$

$$\frac{\partial \mathcal{L}}{\partial r_n} = \left[\sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \left(\sum_j \Omega_j \epsilon_{ijn}\right)\right] - (\boldsymbol{\nabla} U)_n$$

$$\frac{\partial \mathcal{L}}{\partial r_n} = \left[\sum_{ij} m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \, \Omega_j \epsilon_{ijn}\right] - (\boldsymbol{\nabla} U)_n$$

Then using $\sum_{ij} A_i B_j \epsilon_{ijn} = (\boldsymbol{A} \times \boldsymbol{B})_n$ we rewrite this as a cross-product:

$$\frac{\partial \mathcal{L}}{\partial r_n} = m\left[(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}\right]_n - (\boldsymbol{\nabla} U)_n$$

and then distribute

$$\frac{\partial \mathcal{L}}{\partial r_n} = m(\boldsymbol{v} \times \Omega)_n + m((\boldsymbol{\Omega} \times \boldsymbol{r}) \times \Omega)_n - (\boldsymbol{\nabla} U)_n$$

Now let's go back and differentiate $\mathcal{L}$ w.r.t. $\dot{r}_n$ (dropping the potential term since $\partial U/\partial \dot{r}_n = 0$)

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_n} = \sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \frac{\partial}{\partial \dot{r}_n}(\dot{r}_i + (\boldsymbol{\Omega} \times \boldsymbol{r})_i)$$

Then use $\partial \dot{r}_i / \partial \dot{r}_n = \delta_{in}$ and $\partial r_i / \partial \dot{r}_n = 0$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_n} = \sum_i m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_i \, \delta_{in} = m(\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})_n = mv_n + m(\boldsymbol{\Omega} \times \boldsymbol{r})_n$$

Now take the time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{r}_n} = ma_n + m(\dot{\boldsymbol{\Omega}} \times \boldsymbol{r})_n + m(\boldsymbol{\Omega} \times \boldsymbol{v})_n$$

So the Lagrange equation of motion for component r_n reads

$$ma_n + m(\dot{\boldsymbol{\Omega}} \times \boldsymbol{r})_n + m(\boldsymbol{\Omega} \times \boldsymbol{v})_n = m(\boldsymbol{v} \times \Omega)_n + m((\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega})_n - (\boldsymbol{\nabla} U)_n$$

Combining the components into vectors,

$$m\boldsymbol{a} + m\dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + m\boldsymbol{\Omega} \times \boldsymbol{v} = m\boldsymbol{v} \times \Omega + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} - \boldsymbol{\nabla} U$$

Then permute to flip signs, and use $\boldsymbol{F} = -\boldsymbol{\nabla} U$

$$m\boldsymbol{a} - m\boldsymbol{r} \times \dot{\boldsymbol{\Omega}} - m\boldsymbol{v} \times \boldsymbol{\Omega} = m\boldsymbol{v} \times \Omega + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} + \boldsymbol{F}$$

and rearrange to get vector sum of real force and the three pseudo-forces: “azimuthal” (a.k.a. Euler) force, Coriolis force, and centrifugal force.

$$m\boldsymbol{a} = \boldsymbol{F} + m\,\boldsymbol{r} \times \dot{\boldsymbol{\Omega}} + 2m\,\boldsymbol{v} \times \boldsymbol{\Omega} + m\,(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

`http://positron.hep.upenn.edu/p351/files/0302_pseudoforce.pdf`

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- Read the rest of Ch10 (and answer online questions) during spring break, or as soon as you get back.
- Enjoy your week off. Safe travels!