Physics 351 — Monday, March 12, 2018

- HW7 due this Friday, 3/16. Handout is in back of room. (I put it online last Friday.) It is mostly chapter 9, a bit of chapter 8, and some introductory stuff from chapter 10.
- Midterm (3/26) will cover only ch 7,8,9.
- Today we'll start Chapter 10, which you've now read.

Under what circumstances does the angular momentum vector $\boldsymbol{L}$ of a rotating rigid body point in the same direction as the rotation vector $\boldsymbol{\Omega}$? (Or perhaps they always point in the same direction?) How does this relate to having properly balanced (symmetrized) wheels on your car? What would the (off-diagonal) products of inertia be for a well-balanced (i.e. axially symmetric) wheel, for rotation about its axle?

If a body has an axis of rotational symmetry, what can you say about the body's principal axes (of inertia)? What if a body has two perpendicular planes of reflection symmetry through the origin? Does a body having no particular symmetry still have principal axes?

Find the mistake in the following reasoning: "Since the body appears to be at rest to the observer in the body frame, the body's angular momentum $\boldsymbol{L}$ in the body frame must be zero." This is an unusual question, in that Taylor doesn't directly answer it. You have to think about what Taylor means when he talks about the $\boldsymbol{L}$ vector as observed from (not "calculated from") the body frame. What is the body frame, anyway, and how does it relate to the space frame?

We've seen several illustrations now (most recently the Kepler problem) of the usefulness of considering the motion of the CoM separately from the motion of consituents w.r.t. the CoM.

For instance, the "falling chain" problem from HW2 made use of the incredibly useful result

$$M_{\text{total}}\ddot{\boldsymbol{R}}_{\text{cm}} = \sum \boldsymbol{F}_{\text{external}}$$

where the CM motion depends only on the external forces, even if the the consituent parts move/interact in a non-trivial way.

Separating out the CM motion is similarly useful for describing rotation.

Angular momentum of particle α w.r.t. some origin O is

$$\ell_\alpha = \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha = \boldsymbol{r}_\alpha \times m\boldsymbol{v}_\alpha$$

Since angular momentum is an "extensive" (additive) quantity (as is generally true of conserved quantities), a system of particles has angular momentum

$$\boldsymbol{L} = \sum_\alpha \boldsymbol{r}_\alpha \times m\boldsymbol{v}_\alpha = \cdots = \boldsymbol{R}_{\rm cm} \times M_{\rm total}\boldsymbol{V}_{\rm cm} + \sum_\alpha r'_\alpha \times m_\alpha \boldsymbol{v}'_\alpha$$

where $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{R}_{\rm cm}$ and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}_{\rm cm}$. So we can separate

$$\boldsymbol{L} = \boldsymbol{L}_{\rm (motion\ of\ CM)} + \boldsymbol{L}_{\rm (motion\ wrt\ CM)} = \boldsymbol{L}_{\rm orbital} + \boldsymbol{L}_{\rm spin}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_{\rm orbital} = \boldsymbol{R}_{\rm cm} \times \boldsymbol{F}_{\rm external}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_{\rm spin} = \sum_\alpha \boldsymbol{r}'_\alpha \times \boldsymbol{F}^{\rm external}_{\rm on\ \alpha} \equiv \boldsymbol{\tau}_{\rm external}(\text{about CM})$$

If $\boldsymbol{F}^{\rm external}$ acts through CM, then $\boldsymbol{L}_{\rm spin} = \text{const}$ (flying hammer)

Kinetic energy for a system of particles is

$$T = \sum_\alpha \frac{1}{2} m_\alpha \boldsymbol{v}_\alpha^2$$

We can again subtract out some common $\boldsymbol{V}$ (at this stage $\boldsymbol{V}$ needn't be $\boldsymbol{V}_{\rm cm}$ — this is just math) from each $\boldsymbol{v}_\alpha$ and define

$$\boldsymbol{v}'_\alpha = \boldsymbol{v}_\alpha - \boldsymbol{V} \qquad \leftrightarrow \qquad \boldsymbol{v}_\alpha = \boldsymbol{V} + \boldsymbol{v}'_\alpha$$

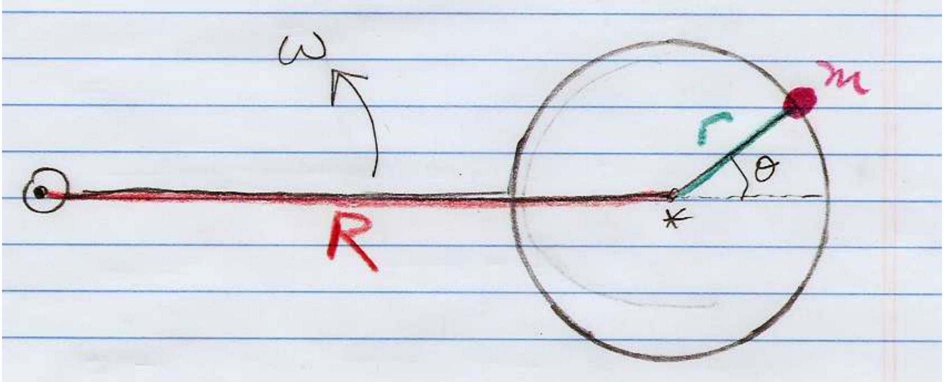
which lets us write

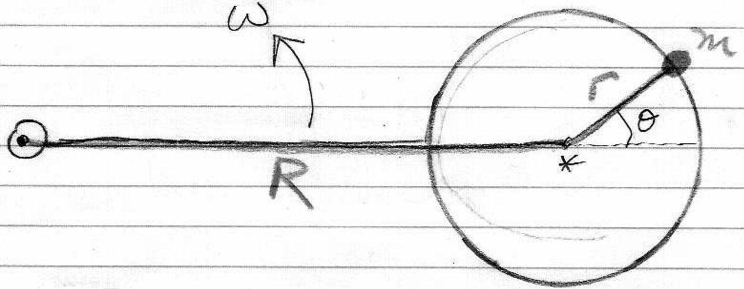
$$\boldsymbol{v}_\alpha^2 = (\boldsymbol{V} + \boldsymbol{v}'_\alpha)^2 = \boldsymbol{V}^2 + (\boldsymbol{v}'_\alpha)^2 + 2\boldsymbol{V} \cdot \boldsymbol{v}'_\alpha$$

which can be extremely handy for writing the K.E. in Lagrangians

$$\boldsymbol{v}_\alpha = \boldsymbol{V} + \boldsymbol{v}'_\alpha \qquad \rightarrow \qquad \boldsymbol{v}_\alpha^2 = \boldsymbol{V}^2 + (\boldsymbol{v}'_\alpha)^2 + 2\boldsymbol{V}\cdot\boldsymbol{v}'_\alpha$$

For instance, write the K.E. of the bead that slides on the rotating massless, frictionless hoop from the scenario of HW5/q4





$\vec{V}_* = (\omega R)$ (points upward in figure)

$\vec{V}_m - \vec{V}_* = (\omega + \dot{\theta}) r$ (points tangentially to hoop)

$$V_m^2 = V_*^2 + |\vec{V}_m - \vec{V}_*|^2 + 2 V_* |\vec{V}_m - \vec{V}_*| \cos\theta$$

$$= (\omega R)^2 + (\omega + \dot{\theta})^2 r^2 + 2 \omega R (\omega + \dot{\theta}) r \cos\theta$$

Kinetic energy for a system of particles is

$$T = \sum_\alpha \frac{1}{2} m_\alpha \boldsymbol{v}_\alpha^2$$

We can subtract out some common $\boldsymbol{V}$ (at this stage $\boldsymbol{V}$ needn't be $\boldsymbol{V}_{\rm cm}$ — this is just math) from each $\boldsymbol{v}_\alpha$ and write

$$\boldsymbol{v}_\alpha^2 = (\boldsymbol{V} + \boldsymbol{v}'_\alpha)^2 = \boldsymbol{V}^2 + (\boldsymbol{v}'_\alpha)^2 + 2\boldsymbol{V} \cdot \boldsymbol{v}'_\alpha$$

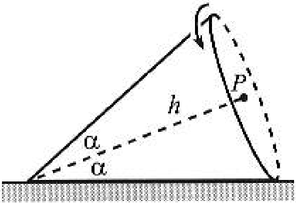
which can also be handy if $\boldsymbol{V} = \dot{\boldsymbol{R}}$, where $\boldsymbol{R}$ is cleverly chosen to be a point which is **instantaneously** at rest, so then

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\boldsymbol{v}'_\alpha)^2$$

i.e. the K.E. can be calculated w.r.t. the point $\boldsymbol{R}$ which is instantaneously at rest, such as the point of contact of a wheel with the road. (HW7/q6.)

To illustrate the result $T = \frac{1}{2}\sum m_\alpha \dot{\mathbf{r}}_\alpha'^{\,2}$ (Eq. 10.18) that the total KE of a body is just the rotational KE relative to any point that is instantaneously at rest, do the following: Write down the KE of a uniform wheel (mass M, radius R) rolling with speed v along a flat road, as the sum of the energies of the CM motion and the rotation about the CM. Now instead write it as the energy of rotation about the instantaneous point of contact with the road and show that you get the same answer. (Recall that the energy of rotation is $\frac{1}{2}I\omega^2$, that the moment of inertia of a uniform wheel about its center is $I = \frac{1}{2}MR^2$, and that the moment of inertia of the wheel about a point on the rim is $I' = \frac{3}{2}MR^2$.) [This problem is assigned just to remind you of useful result (Eq. 10.18).]

A non-trivial example for which this idea is useful is calculating the K.E. of a cone that rolls without slipping on a plane. Morin's book works this out, if you're curious. (Or you could try it yourself!)



Kinetic energy for a system of particles is

$$T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \boldsymbol{v}_{\alpha}^2$$

If we actually subtract out $\boldsymbol{V}_{\rm cm}$ from each $\boldsymbol{v}_{\alpha}$ (instead of subtracting out some arbitrary $\boldsymbol{V}$) then

$$\sum_{\alpha} \frac{1}{2} m_{\alpha} \boldsymbol{v}_{\alpha}^2 = \frac{1}{2} \sum_{\alpha} m_{\alpha} (\boldsymbol{V}_{\rm cm} + \boldsymbol{v}'_{\alpha})^2$$

$$= \frac{1}{2} \left(\sum_{\alpha} m_{\alpha} \right) \boldsymbol{V}_{\rm cm}^2 + \sum_{\alpha} \frac{1}{2} \mathrm{m}_{\alpha} (\boldsymbol{v}'_{\alpha})^2 + \boldsymbol{V}_{\rm cm} \cdot \left(\sum_{\alpha} m_{\alpha} \boldsymbol{v}'_{\alpha} \right)$$

$$= \frac{1}{2} M_{\rm total} \boldsymbol{V}_{\rm cm}^2 + \sum_{\alpha} \frac{1}{2} \mathrm{m}_{\alpha} (\boldsymbol{v}'_{\alpha})^2 + 0$$

$$= T_{(\text{motion of CM})} + T_{(\text{motion relative to CM})}$$

e.g. $\frac{1}{2} M V^2 + \frac{1}{2} I \omega^2$ from first-year physics. So now let's set aside the translational motion and focus on the rotational motion.

Consider a body, rotating about the origin O with fixed rotation vector $\mathbf{\Omega}$. Constituent particle α has angular momentum

$$\boldsymbol{\ell}_\alpha = \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha \boldsymbol{v}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha (\mathbf{\Omega} \times \boldsymbol{r}_\alpha)$$

(Pause to ponder the last step here: $\mathbf{\Omega} \times \boldsymbol{r}_\alpha$ is $\mathrm{d}\boldsymbol{r}_\alpha/\mathrm{d}t$ evaluated in the "space" frame, given that $\boldsymbol{r}$ is at a fixed position in the "body" frame. Also illustrate direction of $\boldsymbol{\ell}_\alpha$ for some cases, and ponder whether $\boldsymbol{\ell}_\alpha$ is constant. Also notice that same (e.g. circular) motion, evaluated for different origin, has different $\boldsymbol{\ell}_\alpha$.)

Constituent particle α has angular momentum

$$\boldsymbol{\ell}_\alpha = \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha \boldsymbol{v}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

$$\ell_\alpha = m_\alpha \boldsymbol{r}_\alpha \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

So the rigid body as a whole has angular momentum

$$\boldsymbol{L} = \sum_\alpha m_\alpha \boldsymbol{r}_\alpha \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

Consider each component L_i $(i = x, y, z)$ of $\boldsymbol{L}$.

$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \, (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)_m \, \epsilon_{kmi}$$

$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \left(\sum_{j,n} \Omega_j \, r_{\alpha,n} \, \epsilon_{jnm} \right) \epsilon_{kmi}$$

Digression: "Einstein" notation for linear algebra

Cartesian Einstein notation

vector $\vec{r} = (x,y,z) = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \sum_i r_i \hat{e}_i$

the i^{th} component $(i \in \{1,2,3\})$ of vector $\vec{A}$ is A_i

Kronecker delta : $\delta_{ij} = 1$ if $i=j$, else 0

dot product : $\vec{A}\cdot\vec{B} = \sum_{ij} A_i B_j \delta_{ij} = \sum_i A_i B_i$

matrix · vector : $\underline{\underline{M}}\cdot\vec{r} = \sum_{ij} M_{ij} r_j \hat{e}_i$

$$(\underline{\underline{M}}\cdot\vec{r})_i = \sum_j M_{ij} r_j$$

matrix multiply : $(\underline{\underline{M}}\cdot\underline{\underline{N}})_{ij} = \sum_k M_{ik} N_{kj}$

http://positron.hep.upenn.edu/p351/files/0119_cartesian_einstein.pdf

Levi-Civita symbol (a.k.a. permutation symbol, antisymmetric symbol)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \in \{123, 231, 312\} \\ -1 & \text{if } ijk \in \{213, 321, 132\} \\ 0 & \text{otherwise} \end{cases}$$

So $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

all others are zero.

Cross product: $\vec{A} \times \vec{B} = \sum_{ijk} A_i B_j \hat{e}_k \, \varepsilon_{ijk}$

$$\vec{A} \times \vec{B} = (A_1 B_2 - A_2 B_1)\hat{e}_3 + (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$$

$$(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$$

You can eliminate **sum over repeated index** k using this identity:

Incredibly useful identity:

$$\varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

Now use it to prove the dreaded "BAC−CAB" rule:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \sum_{ijk} A_i\,(\vec{B}\times\vec{C})_j\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklmn} A_i\,(B_l\,C_m\,\hat{e}_n\,\varepsilon_{lmn})_j\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i\,(B_l\,C_m\,\varepsilon_{lmj})\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i\,B_l\,C_m\,\hat{e}_k\,(\varepsilon_{lmj}\,\varepsilon_{ijk})$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k \left(\varepsilon_{lmj} \varepsilon_{ijk}\right)$$

$$\varepsilon_{lmj} \varepsilon_{ijk} = \varepsilon_{lmj} \varepsilon_{kij} = \left(\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}\right)$$

$$= \sum_{ijklm} \left(A_i B_l C_m \hat{e}_k \delta_{lk}\delta_{mi} - A_i B_l C_m \hat{e}_k \delta_{li}\delta_{mk}\right)$$

$$= \sum_{ik} \left(A_i B_k C_i \hat{e}_k - A_i B_i C_k \hat{e}_k\right)$$

$$= \left(\sum_i A_i C_i\right)\left(\sum_k B_k \hat{e}_k\right) - \left(\sum_i A_i B_i\right)\left(\sum_k C_k \hat{e}_k\right)$$

$$= \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

$$= \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

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$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \left(\sum_{j,n} \Omega_j \, r_{\alpha,n} \, \epsilon_{jnm} \right) \epsilon_{kmi}$$

Still working out component L_i of $\boldsymbol{L}$ $\quad (i = x, y, z)$ . . .

$$L_i = \sum_{\alpha,k,m} m_\alpha\, r_{\alpha,k} \left(\sum_{j,n} \Omega_j\, r_{\alpha,n}\, \epsilon_{jnm} \right) \epsilon_{kmi}$$

$$L_i = \sum_{\alpha,j,k,m,n} m_\alpha\, r_{\alpha,k}\, r_{\alpha,n}\, \epsilon_{jnm}\, \epsilon_{ikm}\, \Omega_j$$

$$L_i = \sum_j \left(\sum_{\alpha,k,m,n} m_\alpha\, r_{\alpha,k}\, r_{\alpha,n}\, \epsilon_{jnm}\, \epsilon_{ikm} \right) \Omega_j \equiv \sum_j I_{ij}\, \Omega_j$$

is a linear (i.e. matrix multiplication) relationship between the two vectors $\boldsymbol{L}$ and $\boldsymbol{\Omega}$. The moment-of-inertia tensor $\boldsymbol{I}$ has components

$$I_{ij} = \sum_{\alpha,k,m,n} m_\alpha\, r_{\alpha,k}\, r_{\alpha,n}\, \epsilon_{jnm}\, \epsilon_{ikm} = \sum_{\alpha,k,m,n} m_\alpha\, r_{\alpha,k}\, r_{\alpha,n}\, (\delta_{ij}\delta_{kn} - \delta_{in}\delta_{jk})$$

$$I_{ij} = \sum_\alpha m_\alpha \left[\left(\sum_k r_{\alpha,k}^2\, \delta_{ij} \right) - r_{\alpha,i}\, r_{\alpha,j} \right] = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

So vectors $\boldsymbol{L}$ and $\boldsymbol{\Omega}$ are related by a matrix multiplication,

$$\underline{\boldsymbol{L}} = \underline{\underline{\boldsymbol{I}}}\ \underline{\boldsymbol{\Omega}}$$

where $\underline{\underline{\boldsymbol{I}}}$ is a real, symmetric matrix with components

$$I_{ij} = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

which implies that its eigenvalues are real and that $\underline{\underline{\boldsymbol{I}}}$ can be diagonalized by an orthogonal matrix $\underline{\underline{\boldsymbol{R}}}$, meaning that there exists an orthogonal matrix (in fact a rotation matrix) $\underline{\underline{\boldsymbol{R}}}$ such that

$$\boldsymbol{R}\ \boldsymbol{I}\ \boldsymbol{R}^T$$

is diagonal. In other words, you can rotate into a basis in which $\boldsymbol{I}$ is diagonal.

Anyway, let's try writing down the components of $\boldsymbol{I}$. (Write down I_{xx}, I_{xy}, I_{xz}, etc.)

If you're stranded on a delayed airplane flight (with no internet!) and you desperately need to remember how to write down the moment-of-inertia tensor (whose off-diagonal elements I have trouble remembering), now you know that it's not so bad:

- Remember $\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\Omega}$ $\qquad\qquad$ $\boldsymbol{L} = \sum \boldsymbol{\ell}$ of constituents
- Start with $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$
- Use $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \quad \rightarrow \quad \boldsymbol{\ell} = m\,\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$
- Work out the linear relationship between ℓ_i and Ω_j, e.g. explicitly writing out ℓ_z.

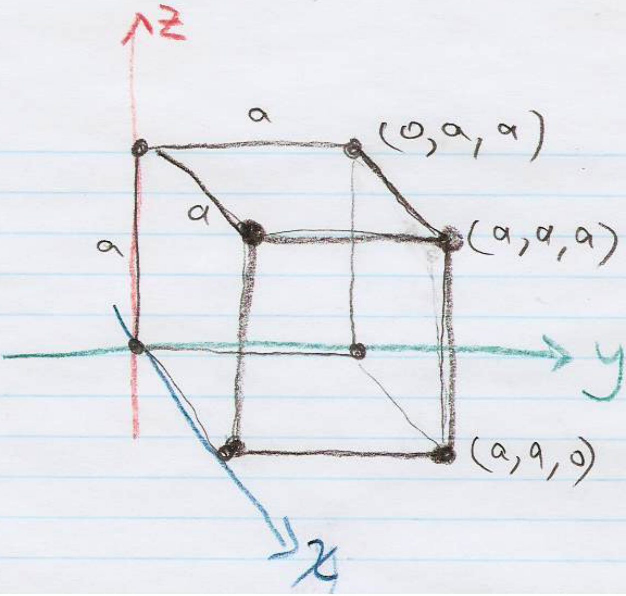
Let's do that explicitly, using conventional vector notation.

The concise way to write $\boldsymbol{I}$, which makes its symmetry obvious.

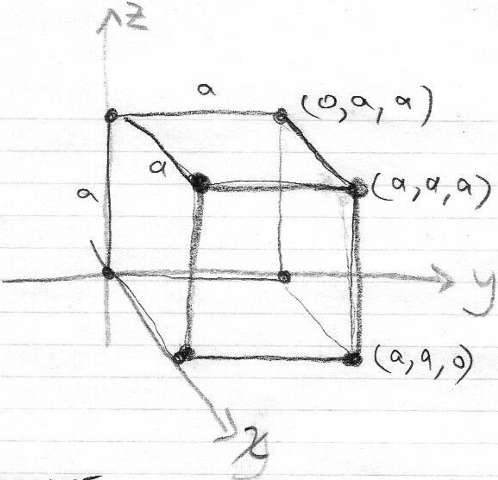
$$I_{ij} = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

And here's a problem from next week's HW8 to practice calculating $\boldsymbol{I}$ for some simple cases:

A rigid body comprises 8 equal masses m at the corners of a cube of side a, held together by massless struts. (a) Use the definitions (Eq. 10.37 and 10.38) $I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2)$ and $I_{xy} = -\sum m_\alpha x_\alpha y_\alpha$ (and cyclic permutations) to find the moment of inertia tensor $\boldsymbol{I}$ for rotation about a corner O of the cube. (Use axes along the three edges through O.) (b) Find the inertia tensor of the same body but for rotation about the center of the cube. (Again use axes parallel to the edges.) Explain why in this case certain elements of $\boldsymbol{I}$ could be expected to be zero.



0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1



0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1

(a) About corner

$$I_{xx} = \sum m(y^2+z^2) = ma^2(4+2\times2) = 8ma^2$$

$$I_{xy} = -\sum mxy = -ma^2(2) = -2ma^2$$

$$\underline{\underline{I}} = ma^2 \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

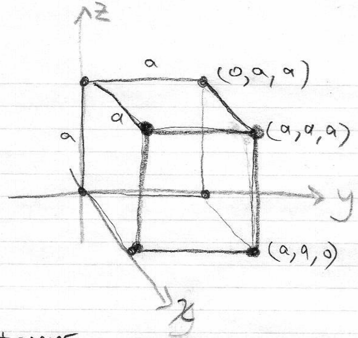
D) About center

$$(x, y, z) = \left(\frac{a}{2}\right)\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$I_{xx} = \frac{ma^2}{4}(8 \times 2) = 4ma^2$$

$$I_{xy} = \frac{ma^2}{4}(4 + (-4)) = 0$$

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} 4ma^2$$



0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

(a) About corner

$$I_{xx} = \sum m(y^2+z^2) = ma^2(4+2\times 2) = 8ma^2$$

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$$\underline{\underline{I}} = ma^2\begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

The first calculation (part a) put origin at corner of cube. If we rotate about $\hat{\boldsymbol{x}}$, will $\boldsymbol{L}$ and $\boldsymbol{\Omega}$ be parallel?

What if we rotate about axis that goes from origin to far opposite diagonal corner?

The first calculation (part a) put origin at corner of cube. Using that origin, let's rotate about axis from origin to far opposite diagonal corner:

```
$ math
Mathematica 10.0 for Linux x86 (64-bit)
Copyright 1988-2014 Wolfram Research, Inc.

In[1]:= {{8,-2,-2},{-2,8,-2},{-2,-2,8}} . {1,1,1}


Out[1]= {4, 4, 4}

In[2]:=
```

One way you could predict that this would be true is that this corner-to-corner axis passes through the center of the cube, and we know that since cube's symmetry (about its center) gives us 3 degenerate eigenvalues, any axis passing through the cube's center should be a principal axis.

Once you know how to calculate $\underline{\underline{\boldsymbol{I}}}$, you can write the angular momentum

$$\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\omega}$$

and the kinetic energy

$$T = \frac{1}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{L} = \frac{1}{2}\,\boldsymbol{\omega}\cdot(\underline{\underline{\boldsymbol{I}}}\boldsymbol{\omega})$$

which generalize the freshman physics results

$$L = I\omega \qquad\qquad T = \frac{1}{2}\,I\omega^2$$

If we rotate coordinate axes into basis in which $\underline{\underline{\boldsymbol{I}}}$ is diagonal, then

$$T = \frac{1}{2}\,(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \lambda_3\Omega_3^2) \qquad \boldsymbol{L} = (\lambda_1\Omega_1, \lambda_2\Omega_2, \lambda_3\Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{\boldsymbol{I}}}$ (i.e. are the 3 principal moments of inertia). So life is simpler in the "principal axes" basis.

If we rotate coordinate axes into basis in which $\underline{\underline{\boldsymbol{I}}}$ is diagonal, then

$$T = \frac{1}{2}\left(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \lambda_3\Omega_3^2\right) \qquad \boldsymbol{L} = (\lambda_1\Omega_1, \lambda_2\Omega_2, \lambda_3\Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{\boldsymbol{I}}}$ (i.e. are the 3 principal moments of inertia).

Math fact: Given a real symmetric 3×3 matrix, $\underline{\underline{\boldsymbol{I}}}$, there exist three orthonormal real vectors $\boldsymbol{e}_i$ such that

$$\underline{\underline{\boldsymbol{I}}}\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$$

The unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ (the eigenvectors of $\underline{\underline{\boldsymbol{I}}}$) are called the principal axes of the rigid body. In most cases of interest, you can find the principal axes by symmetry, instead of having to solve the eigenvalue/eigenvector problem.

With $\underline{\underline{\boldsymbol{I}}} = \int \mathrm{d}m \begin{pmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{pmatrix}$, we get

$$\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\omega} \qquad\qquad T = \frac{1}{2}\,\boldsymbol{\omega}\cdot(\underline{\underline{\boldsymbol{I}}}\boldsymbol{\omega})$$

which generalize the familiar $L = I\omega$ and $T = \frac{1}{2}\,I\omega^2$.

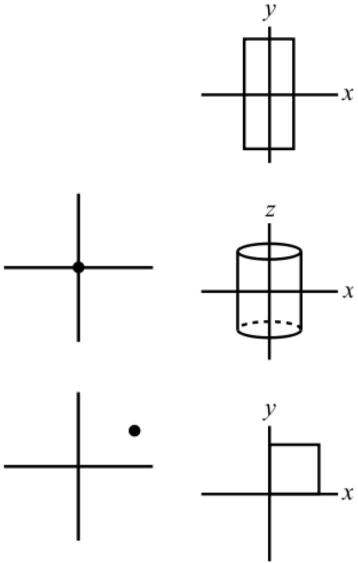
"Principal axes" basis simplifies these expressions considerably:

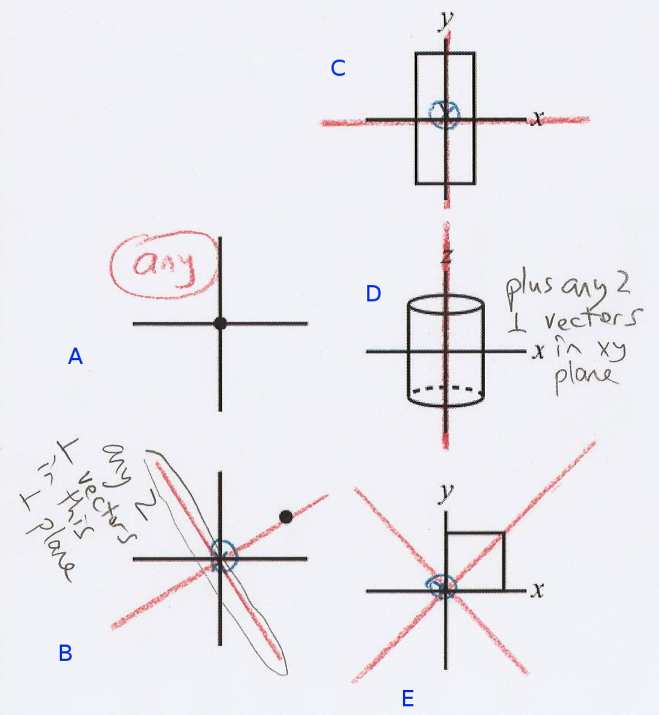
$$\underline{\underline{\boldsymbol{I}}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = \frac{1}{2}\,(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2) \qquad \boldsymbol{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

So it's quite helpful to identify and use principal axes.

Using symmetry, what are the three principal axes of these five objects w.r.t. the origin of the coordinate axes shown? (Your principal axes must pass through the chosen origin.) Note that the left two are point masses in the xy plane.





(A) Any axes.

(B) Axis through point; any axes $\perp$ to this.

(C) x, y, z axes.

(D) z axis; any axes in xy plane.

(E) z axis; axis through CM; axis $\perp$ to this.

Example 1: Point mass at the origin.

principal axes: any axes.

Example 2: Point mass at the point (x_0, y_0, z_0).

principal axes: axis through point, any axes perpendicular to this.

Example 3: Rectangle centered at the origin, as shown.

principal axes: z-axis, axes parallel to sides.

Example 4: Cylinder with axis as z-axis.

principal axes: z-axis, any axes in x-y plane.

Example 5: Square with one corner at origin, as shown.

principal axes: z axis, axis through CM, axis perp to this.

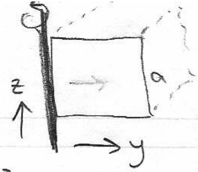
Let's first work through a freshman-physics-like collision problem that involves angular-momentum conservation. Then we'll work through a similar but trickier problem that requires us to project the motion onto the principal axes.

Taylor 10.16
(a) First show that the moment of inertia of a uniform cube of side a and mass M, rotating about an edge, is $(2/3)Ma^2$.

The cube is sliding with velocity $\boldsymbol{v}$ across a flat horizontal frictionless table when it hits a tiny step ($\perp \boldsymbol{v}$), and the leading lower edge comes abruptly to rest.

(b) Find the cube's angular velocity just after the collision.

(c) Find the minimum speed v for which the cube rolls over after hitting the step. (Actually just write down an equation for the minimum speed — the algebra is unenlightening.)



(a)

$$M = \rho a^3$$

$$I_{zz} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho\, (x^2+y^2)$$

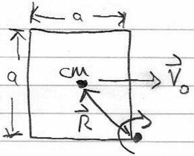
$$= \int_0^a dx \int_0^a dy\; a\rho(x^2+y^2) = a\rho \int_0^a dx \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{a}$$

$$= a\rho \int_0^a dx\, \left(x^2 a + \tfrac{1}{3} a^3\right) = a\rho \left[a\frac{x^3}{3} + \tfrac{1}{3} a^3 x \right]_{x=0}^{a}$$

$$= a\rho\left(\frac{a^4}{3} + \frac{a^4}{3}\right) = \frac{2}{3}\rho a^5 = \frac{2}{3}\left(\frac{M}{a^3}\right)\rho a^5$$

$$= \frac{2}{3} M a^2$$

(b)



$$L_i = \left|\vec{R}_{cm} \times M\vec{V}_0\right| = M\frac{a}{2}V_0$$

$$L_f = I\omega_f = \frac{2}{3}Ma^2\omega_f$$

Angular momentum conservation w.r.t. chosen origin (at fixed position of step)

$$\omega_f = \frac{\left(M\frac{a}{2}V_0\right)}{\frac{2}{3}Ma^2} = \frac{3V_0}{4a}$$

(c)

Energy conservation after collision

$$\frac{1}{2}I\omega_f^2 + Mg\,y_{cm} = Mg\,y_{cm}$$

(just after collision) (when cm is above pivot)

$$\frac{1}{2}\left(\frac{2}{3}Ma^2\right)\left(\frac{M\frac{a}{2}V_0}{\frac{2}{3}Ma^2}\right)^2 + Mg\frac{a}{2} = Mg\frac{a}{\sqrt{2}}$$

after some unenlightening algebra,

$$(V_0)^2_{\text{minimum}} = ag\left(\frac{8}{3}\right)\left(\sqrt{2}-1\right) \simeq 1.1ag$$

Morin Exerise 9.38.

Fig. 9.56

9.38. Striking a triangle **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow? z out of page

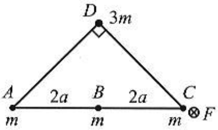
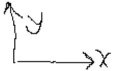


Fig. 9.57



Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

What is the post-impact motion of the CM?

What are the principal axes/moments?

Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.

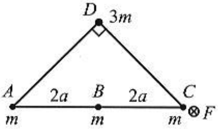
Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\text{CM}}$.

Morin Exerise 9.38.

Fig. 9.56

9.38. Striking a triangle **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow? z out of page



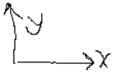


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

Halfway between B and D.

What is the post-impact motion of the CM?

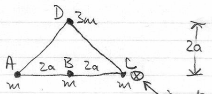
$\boldsymbol{V}_{\mathrm{cm}} = -P/(6m)\hat{\boldsymbol{z}}$ (and stays that way)

What are the principal axes/moments (w.r.t. CM)?

$\lambda_1 = 6ma^2$ $(\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}})$, $\lambda_2 = 8ma^2$ $(\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}})$, $\lambda_3 = 14ma^2$ $(\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}})$

Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.

Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\mathrm{CM}}$.



$D = 3m$ @ $(0, +a, 0)$

put origin at CM

$A = m$ @ $(-2a, -a, 0)$

$B = m$ @ $(0, -a, 0)$

$C = m$ @ $(+2a, -a, 0)$

impulse at point C $= P = \int F\,dt$ into page

Immediately after impulsive blow $-P\hat{z}$ to C,

$$\vec{P}_{cm} = -P\hat{z} \Rightarrow \vec{V}_{cm} = -\frac{P}{6m}\hat{z}$$

$$\vec{L}_{\substack{\text{relative}\\ \text{to cm}}} = \vec{r} \times \vec{p} = (+2a, -a, 0) \times (0, 0, -P)$$

$$= (yP_z - zP_y,\ zP_x - xP_z,\ xP_y - yP_x)$$

$$= (aP, 2aP, 0) = aP(1, 2, 0)$$

Find principal moments:

confirm hunch that $I_{xy} = 0$: $I_{xy} = -\sum mxy = -ma^2(-2+2) = 0$

$$I_{xx} = \sum m(y^2 + z^2) = \sum my^2 = ma^2(3+3) = 6ma^2$$

$$I_{yy} = \sum m(x^2 + z^2) = \sum mx^2 = ma^2(4+4) = 8ma^2$$

$$I_{zz} = \sum m(x^2 + y^2) = 14ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix} ma^2$$

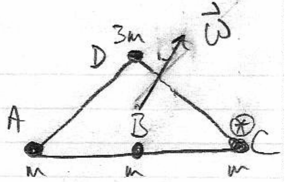
conveniently enough! so $\hat{x}, \hat{y}, \hat{z}$ are (initially) principal axes.

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$$\Rightarrow \omega_1 = \frac{aP}{6ma^2} \qquad \omega_2 = \frac{2aP}{8ma^2} \qquad \omega_3 = 0$$

$$\vec{\omega}_{\substack{\text{immediately}\\ \text{after impact}}} = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right)$$

$$\vec{V}_{\substack{\text{immediately}\\ \text{after}\\ \text{impact}}} = \vec{V}_{cm} + \vec{\omega}\times\vec{r}$$



$$\vec{V}_{cm} = \frac{P}{m}\left(0,0,-\frac{1}{6}\right)$$

$$\vec{\omega}\times\vec{r}_A = \frac{P}{ma}\left(\frac{1}{6},\frac{1}{4},0\right)\times(-2a,-a,0) = \frac{P}{m}\left(0,0,\frac{1}{3}\right)$$

$$\vec{\omega}\times\vec{r}_B = \frac{P}{ma}\left(\frac{1}{6},\frac{1}{4},0\right)\times(0,-a,0) = \frac{P}{m}\left(0,0,-\frac{1}{6}\right)$$

$$\vec{\omega}\times\vec{r}_C = \frac{P}{ma}\left(\frac{1}{6},\frac{1}{4},0\right)\times(+2a,-a,0) = \frac{P}{m}\left(0,0,-\frac{2}{3}\right)$$

$$\vec{\omega}\times\vec{r}_D = \frac{P}{ma}\left(\frac{1}{6},\frac{1}{4},0\right)\times(0,+a,0) = \frac{P}{m}\left(0,0,\frac{1}{6}\right)$$

$$\Rightarrow \quad \vec{V}_A = \frac{P}{m}\left(0,0,\frac{1}{6}\right) \qquad \vec{V}_C = \frac{P}{m}\left(0,0,-\frac{5}{6}\right)$$

$$\vec{V}_B = \frac{P}{m}\left(0,0,-\frac{1}{3}\right) \qquad \vec{V}_D = \frac{P}{m}(0,0,0)$$

In[9]:=

```
{1 / 6, 1 / 4, 0} × {-2, -1, 0}
```

Out[9]= $\left\{0,\ 0,\ \frac{1}{3}\right\}$

In[10]:=

```
{1 / 6, 1 / 4, 0} × {0, -1, 0}
```

Out[10]=

$$\left\{0,\ 0,\ -\frac{1}{6}\right\}$$

In[11]:=

```
{1 / 6, 1 / 4, 0} × {+2, -1, 0}
```

Out[11]=

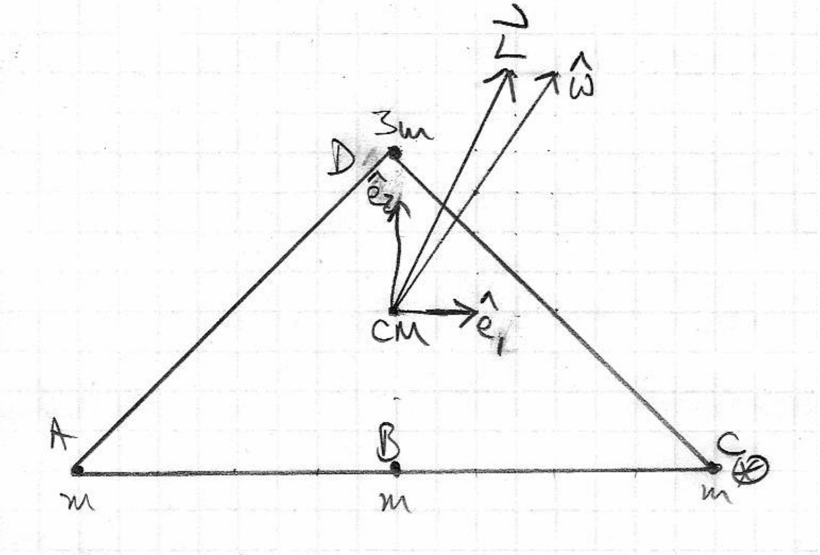
$$\left\{0,\ 0,\ -\frac{2}{3}\right\}$$

In[12]:=

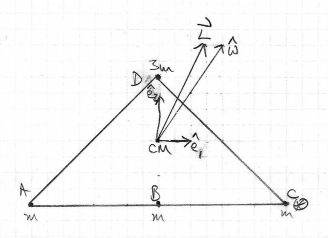
```
{1 / 6, 1 / 4, 0} × {0, +1, 0}
```

Out[12]=

$$\left\{0,\ 0,\ \frac{1}{6}\right\}$$



What will the subsequently happen to $\boldsymbol{V}_{\rm cm}$? To $\boldsymbol{L}$? To $\boldsymbol{\omega}$? To the orientations of the principal axes? With no applied torque, how does $\boldsymbol{\omega}$ evolve in time?



$\lambda_1 = 6ma^2,\ \lambda_2 = 8ma^2,\ \lambda_3 = 14ma^2.$

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1, 2, 0)$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3\omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Try other initial ω vectors:

https://www.youtube.com/watch?v=dVhGyxkBKzI
https://www.youtube.com/watch?v=4Ntgvun8GuY
https://www.youtube.com/watch?v=YKSEu_c3YdY

(For zero torque)

$$0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{space}\\ \text{axes}}} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{body}\\ \text{axes}}} + \vec{\omega} \times \vec{L}$$

$$(\dot{\omega}_1\lambda_1, \dot{\omega}_2\lambda_2, \dot{\omega}_3\lambda_3) = -\vec{\omega} \times (\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3)$$

$$= -(\omega_2\omega_3\lambda_3 - \omega_3\omega_2\lambda_2, \omega_3\omega_1\lambda_1 - \omega_1\omega_3\lambda_3, \omega_1\omega_2\lambda_2 - \omega_2\omega_1\lambda_1)$$

$$= (\omega_2\omega_3(\lambda_2 - \lambda_3), \omega_1\omega_3(\lambda_3 - \lambda_1), \omega_1\omega_2(\lambda_1 - \lambda_2))$$

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{(\lambda_2 - \lambda_3)}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\dot{\omega}_2 = \omega_1\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

It's fun to consider e.g. $\lambda_3 > \lambda_2 > \lambda_1$ for tossed book.

$\lambda_3 > \lambda_2 > \lambda_1$. Start out e.g. about $\hat{e}_3$, with ω_1 and ω_2 both $\ll \omega_3$.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\ddot{\omega}_1 \simeq \dot{\omega}_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \underbrace{\omega_1 \omega_2}_{\text{"small × small"}} \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$\Rightarrow \omega_3 \approx$ constant (varies very slowly)

$$\ddot{\omega}_1 = \omega_1 \left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)\left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right) = -\omega_1 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq \dot{\omega}_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = \omega_2 \left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right)\left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq -\omega_2 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\right) \equiv -\Omega^2 \omega_2$$

Start out about $\hat{e}_1$ $(\lambda_1 < \lambda_2 < \lambda_3)$, so ω_2 and ω_3 both $\ll \omega_1$ initially.

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \sim (\text{small})^2$$

$$\ddot{\omega}_2 \simeq \omega_1 \dot{\omega}_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = -\omega_2 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_2$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_3 \simeq \omega_1 \dot{\omega}_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = -\omega_3 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) \equiv -\Omega^2 \omega_3$$

Start out about $\hat{e}_2$, so ω_1 and $\omega_3 \ll \omega_2$ initially.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \sim (\text{small})^2$$

$$\ddot{\omega}_1 \simeq \omega_2 \dot{\omega}_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \qquad \ddot{\omega}_3 \simeq \dot{\omega}_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_1 \simeq \omega_2 \left(\omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}\right) \frac{\lambda_2 - \lambda_3}{\lambda_1} = +\omega_1 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3}\right)$$

$$\ddot{\omega}_3 \simeq \left(\omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right) \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = +\omega_3 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3}\right)$$

$\Rightarrow$ exponential growth of ω_1, ω_3

$\longrightarrow$ initial motion about $\hat{e}_2$ won't stay about $\hat{e}_2$

Physics 351 — Monday, March 12, 2018

- HW7 due this Friday, 3/16. Handout is in back of room. (I put it online last Friday.) It is mostly chapter 9, a bit of chapter 8, and some introductory stuff from chapter 10.
- Midterm (3/26) will cover only ch 7,8,9.
- Today we'll start Chapter 10, which you've now read.