Physics 351 — Wednesday, March 14, 2018

- Chapter 10 is (I think) the most difficult chapter in the book, so we will take our time to go through it slowly.
- HW7 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- On problem 1, you should first find $r(\phi)$ then work from the transformed radial equation (Taylor 8.41), where $u = 1/r$

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2\, u(\phi)^2}\, F$$

Consider a body, rotating about the origin O with fixed rotation vector $\boldsymbol{\Omega}$. Constituent particle α has angular momentum

$$\boldsymbol{\ell}_\alpha = \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha \boldsymbol{v}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha(\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

(Pause to ponder the last step here: $\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha$ is $\mathrm{d}\boldsymbol{r}_\alpha/\mathrm{d}t$ evaluated in the "space" frame, given that $\boldsymbol{r}$ is at a fixed position in the "body" frame. Also illustrate direction of $\boldsymbol{\ell}_\alpha$ for some cases, and ponder whether $\boldsymbol{\ell}_\alpha$ is constant. Also notice that same (e.g. circular) motion, evaluated for different origin, has different $\boldsymbol{\ell}_\alpha$.)

Constituent particle α has angular momentum

$$\boldsymbol{\ell}_\alpha = \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha \boldsymbol{v}_\alpha = \boldsymbol{r}_\alpha \times m_\alpha (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

$$\ell_\alpha = m_\alpha \boldsymbol{r}_\alpha \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

So the rigid body as a whole has angular momentum

$$\boldsymbol{L} = \sum_\alpha m_\alpha \boldsymbol{r}_\alpha \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)$$

Consider each component L_i $(i = x, y, z)$ of $\boldsymbol{L}$.

$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \, (\boldsymbol{\Omega} \times \boldsymbol{r}_\alpha)_m \, \epsilon_{kmi}$$

$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \left(\sum_{j,n} \Omega_j \, r_{\alpha,n} \, \epsilon_{jnm} \right) \epsilon_{kmi}$$

Digression: "Einstein" notation for linear algebra

<u>Cartesian Einstein notation</u>

vector $\vec{r} = (x,y,z) = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \sum_i r_i \hat{e}_i$

the i^{th} component $(i \in \{1,2,3\})$ of vector $\vec{A}$ is A_i

Kronecker delta : $\delta_{ij} = 1$ if $i=j$, else 0

dot product : $\vec{A} \cdot \vec{B} = \sum_{ij} A_i B_j \delta_{ij} = \sum_i A_i B_i$

matrix · vector : $\underline{\underline{M}} \cdot \vec{r} = \sum_{ij} M_{ij} r_j \hat{e}_i$

$$(\underline{\underline{M}} \cdot \vec{r})_i = \sum_j M_{ij} r_j$$

matrix multiply : $(\underline{\underline{M}} \cdot \underline{\underline{N}})_{ij} = \sum_k M_{ik} N_{kj}$

http://positron.hep.upenn.edu/p351/files/0119_cartesian_einstein.pdf

Levi-Civita symbol (a.k.a. permutation symbol, antisymmetric symbol)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \in \{123, 231, 312\} \\ -1 & \text{if } ijk \in \{213, 321, 132\} \\ 0 & \text{otherwise} \end{cases}$$

So $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$
all others are zero.

Cross product: $\vec{A} \times \vec{B} = \sum_{ijk} A_i B_j \hat{e}_k \varepsilon_{ijk}$

$$\vec{A} \times \vec{B} = (A_1 B_2 - A_2 B_1)\hat{e}_3 + (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$$

$$(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$$

You can eliminate **sum over repeated index** k using this identity:

Incredibly useful identity:

$$\varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

Now use it to prove the dreaded "BAC-CAB" rule:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \sum_{ijk} A_i\,(\vec{B}\times\vec{C})_j\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklmn} A_i\,(B_l\,C_m\,\hat{e}_n\,\varepsilon_{lmn})_j\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i\,(B_l\,C_m\,\varepsilon_{lmj})\,\hat{e}_k\,\varepsilon_{ijk}$$

$$= \sum_{ijklm} A_i\,B_l\,C_m\,\hat{e}_k\,(\varepsilon_{lmj}\,\varepsilon_{ijk})$$

$$= \sum_{ijklm} A_i B_l C_m \hat{e}_k \left(\varepsilon_{lmj}\,\varepsilon_{ijk}\right)$$

$$\varepsilon_{lmj}\,\varepsilon_{ijk} = \varepsilon_{lmj}\,\varepsilon_{kij} = \left(\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}\right)$$

$$= \sum_{ijklm} \left(A_i B_l C_m \hat{e}_k \,\delta_{lk}\delta_{mi} - A_i B_l C_m \hat{e}_k \,\delta_{li}\delta_{mk}\right)$$

$$= \sum_{ik} \left(A_i B_k C_i \hat{e}_k - A_i B_i C_k \hat{e}_k\right)$$

$$= \left(\sum_i A_i C_i\right)\left(\sum_k B_k \hat{e}_k\right) - \left(\sum_i A_i B_i\right)\left(\sum_k C_k \hat{e}_k\right)$$

$$= \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

$$= \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

Constituent particle α has angular momentum

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$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \left(\sum_{j,n} \Omega_j \, r_{\alpha,n} \, \epsilon_{jnm} \right) \epsilon_{kmi}$$

Still working out component L_i of $\boldsymbol{L}$ $\quad (i = x, y, z)$ ...

$$L_i = \sum_{\alpha,k,m} m_\alpha \, r_{\alpha,k} \left(\sum_{j,n} \Omega_j \, r_{\alpha,n} \, \epsilon_{jnm} \right) \epsilon_{kmi}$$

$$L_i = \sum_{\alpha,j,k,m,n} m_\alpha \, r_{\alpha,k} \, r_{\alpha,n} \, \epsilon_{jnm} \, \epsilon_{ikm} \, \Omega_j$$

$$L_i = \sum_j \left(\sum_{\alpha,k,m,n} m_\alpha \, r_{\alpha,k} \, r_{\alpha,n} \, \epsilon_{jnm} \, \epsilon_{ikm} \right) \Omega_j \equiv \sum_j I_{ij} \, \Omega_j$$

is a linear (i.e. matrix multiplication) relationship between the two vectors $\boldsymbol{L}$ and $\boldsymbol{\Omega}$. The moment-of-inertia tensor $\boldsymbol{I}$ has components

$$I_{ij} = \sum_{\alpha,k,m,n} m_\alpha \, r_{\alpha,k} \, r_{\alpha,n} \, \epsilon_{jnm} \, \epsilon_{ikm} = \sum_{\alpha,k,m,n} m_\alpha \, r_{\alpha,k} \, r_{\alpha,n} \, (\delta_{ij}\delta_{kn} - \delta_{in}\delta_{jk})$$

$$I_{ij} = \sum_\alpha m_\alpha \left[\left(\sum_k r_{\alpha,k}^2 \, \delta_{ij} \right) - r_{\alpha,i} \, r_{\alpha,j} \right] = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

So vectors $\boldsymbol{L}$ and $\boldsymbol{\Omega}$ are related by a matrix multiplication,

$$\underline{\boldsymbol{L}} = \underline{\underline{\boldsymbol{I}}}\ \underline{\boldsymbol{\Omega}}$$

where $\underline{\underline{\boldsymbol{I}}}$ is a real, symmetric matrix with components

$$I_{ij} = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

which implies that its eigenvalues are real and that $\underline{\underline{\boldsymbol{I}}}$ can be diagonalized by an orthogonal matrix $\underline{\underline{\boldsymbol{R}}}$, meaning that there exists an orthogonal matrix (in fact a rotation matrix) $\underline{\underline{\boldsymbol{R}}}$ such that

$$\boldsymbol{R}\ \boldsymbol{I}\ \boldsymbol{R}^T$$

is diagonal. In other words, you can rotate into a basis in which $\boldsymbol{I}$ is diagonal.

Anyway, let's try writing down the components of $\boldsymbol{I}$. (Write down I_{xx}, I_{xy}, I_{xz}, etc.)

If you're stranded on a delayed airplane flight (with no internet!) and you desperately need to remember how to write down the moment-of-inertia tensor (whose off-diagonal elements I have trouble remembering), now you know that it's not so bad:

- Remember $\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\Omega}$ $\qquad\qquad \boldsymbol{L} = \sum \boldsymbol{\ell}$ of constituents
- Start with $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$
- Use $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \quad \rightarrow \quad \boldsymbol{\ell} = m\,\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$
- Work out the linear relationship between ℓ_i and Ω_j, e.g. explicitly writing out ℓ_z.

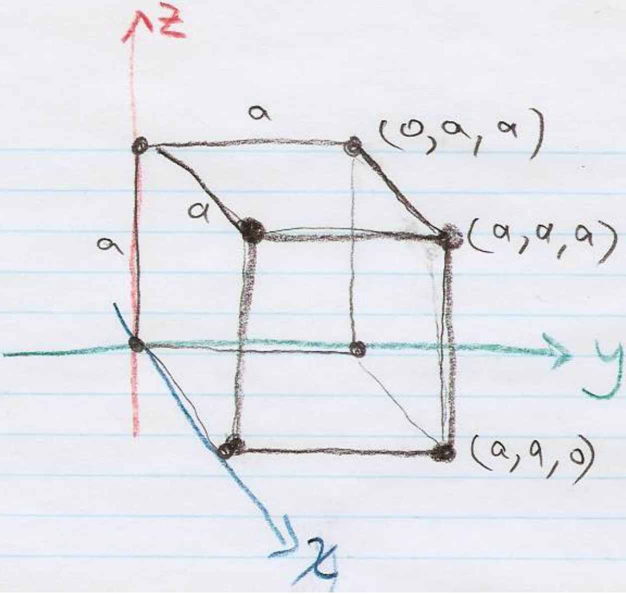
Let's do that explicitly, using conventional vector notation.

The concise way to write $\boldsymbol{I}$, which makes its symmetry obvious.

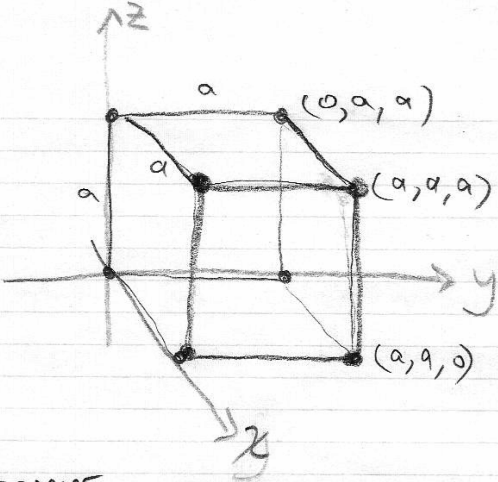
$$I_{ij} = \sum m \left[r^2 \delta_{ij} - r_i r_j \right]$$

And here's a problem from next week's HW8 to practice calculating $\boldsymbol{I}$ for some simple cases:

A rigid body comprises 8 equal masses m at the corners of a cube of side a, held together by massless struts. (a) Use the definitions (Eq. 10.37 and 10.38) $I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2)$ and $I_{xy} = -\sum m_\alpha x_\alpha y_\alpha$ (and cyclic permutations) to find the moment of inertia tensor $\boldsymbol{I}$ for rotation about a corner O of the cube. (Use axes along the three edges through O.) (b) Find the inertia tensor of the same body but for rotation about the center of the cube. (Again use axes parallel to the edges.) Explain why in this case certain elements of $\boldsymbol{I}$ could be expected to be zero.



0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1



0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

(a) About corner

$$I_{xx} = \sum m(y^2+z^2) = ma^2(4+2\times 2) = 8ma^2$$

$$I_{xy} = -\sum m\,xy = -ma^2(2) = -2ma^2$$

$$\underline{\underline{I}} = ma^2\begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

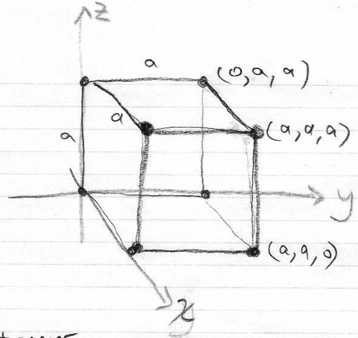
D) About center

$$(x, y, z) = \left(\frac{a}{2}\right)\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$I_{xx} = \frac{ma^2}{4}(8 \times 2) = 4ma^2$$

$$I_{xy} = \frac{ma^2}{4}(4 + (-4)) = 0$$

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} 4ma^2$$



000
001
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111

ⓐ About corner

$$I_{xx} = \sum m(y^2+z^2) = ma^2(4+2\times 2) = 8ma^2$$

$$I_{xy} = -\sum m\,xy = -ma^2(2) = -2ma^2$$

$$\underline{\underline{I}} = ma^2 \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

The first calculation (part a) put origin at corner of cube. If we rotate about $\hat{\boldsymbol{x}}$, will $\boldsymbol{L}$ and $\boldsymbol{\Omega}$ be parallel?

What if we rotate about axis that goes from origin to far opposite diagonal corner?

The first calculation (part a) put origin at corner of cube. Using that origin, let's rotate about axis from origin to far opposite diagonal corner:

```
$ math
Mathematica 10.0 for Linux x86 (64-bit)
Copyright 1988-2014 Wolfram Research, Inc.

In[1]:= {{8,-2,-2},{-2,8,-2},{-2,-2,8}} . {1,1,1}


Out[1]= {4, 4, 4}

In[2]:=
```

One way you could predict that this would be true is that this corner-to-corner axis passes through the center of the cube, and we know that since cube's symmetry (about its center) gives us 3 degenerate eigenvalues, any axis passing through the cube's center should be a principal axis.

Once you know how to calculate $\underline{\underline{\boldsymbol{I}}}$, you can write the angular momentum

$$\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\omega}$$

and the kinetic energy

$$T = \frac{1}{2}\,\boldsymbol{\omega} \cdot \boldsymbol{L} = \frac{1}{2}\,\boldsymbol{\omega} \cdot (\underline{\underline{\boldsymbol{I}}}\boldsymbol{\omega})$$

which generalize the freshman physics results

$$L = I\omega \qquad\qquad T = \frac{1}{2}\,I\omega^2$$

If we rotate coordinate axes into basis in which $\underline{\underline{\boldsymbol{I}}}$ is diagonal, then

$$T = \frac{1}{2}\,(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \lambda_3\Omega_3^2) \qquad \boldsymbol{L} = (\lambda_1\Omega_1, \lambda_2\Omega_2, \lambda_3\Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{\boldsymbol{I}}}$ (i.e. are the 3 principal moments of inertia). So life is simpler in the "principal axes" basis.

If we rotate coordinate axes into basis in which $\underline{\underline{\boldsymbol{I}}}$ is diagonal, then

$$T = \frac{1}{2}\left(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \lambda_3\Omega_3^2\right) \qquad \boldsymbol{L} = (\lambda_1\Omega_1, \lambda_2\Omega_2, \lambda_3\Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{\boldsymbol{I}}}$ (i.e. are the 3 principal moments of inertia).

Math fact: Given a real symmetric 3×3 matrix, $\underline{\underline{\boldsymbol{I}}}$, there exist three orthonormal real vectors $\boldsymbol{e}_i$ such that

$$\underline{\underline{\boldsymbol{I}}}\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$$

The unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ (the eigenvectors of $\underline{\underline{\boldsymbol{I}}}$) are called the principal axes of the rigid body. In most cases of interest, you can find the principal axes by symmetry, instead of having to solve the eigenvalue/eigenvector problem.

With $\underline{\underline{\boldsymbol{I}}} = \int \mathrm{d}m \begin{pmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{pmatrix}$, we get

$$\boldsymbol{L} = \underline{\underline{\boldsymbol{I}}}\,\boldsymbol{\omega} \qquad\qquad T = \frac{1}{2}\,\boldsymbol{\omega}\cdot(\underline{\underline{\boldsymbol{I}}}\boldsymbol{\omega})$$

which generalize the familiar $L = I\omega$ and $T = \frac{1}{2}\,I\omega^2$.

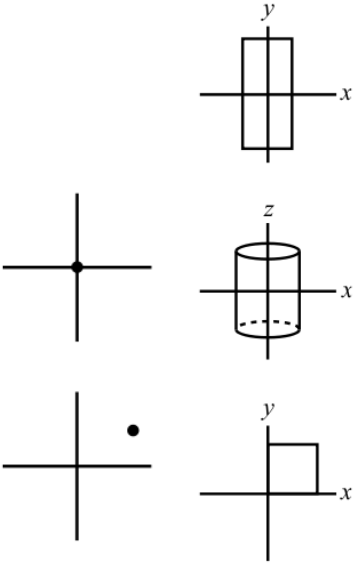
"Principal axes" basis simplifies these expressions considerably:

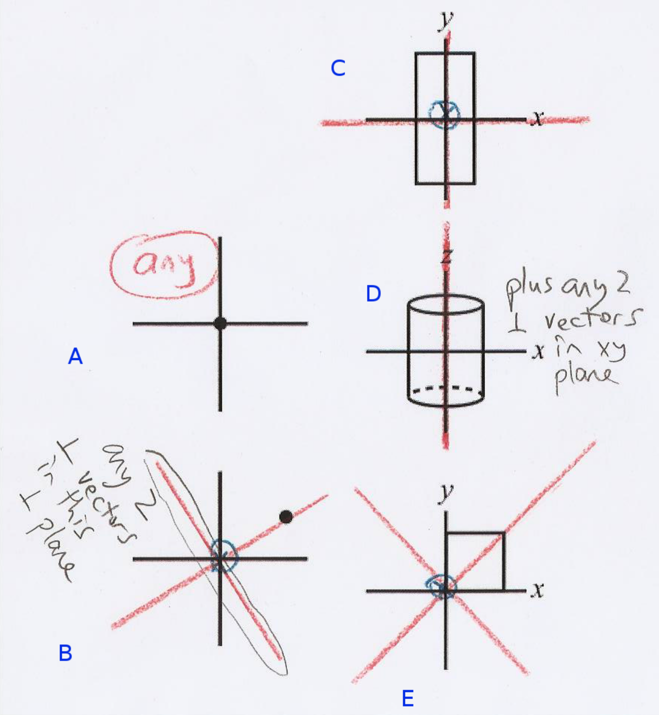
$$\underline{\underline{\boldsymbol{I}}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = \frac{1}{2}\,(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2) \qquad \boldsymbol{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

So it's quite helpful to identify and use principal axes.

Using symmetry, what are the three principal axes of these five objects w.r.t. the origin of the coordinate axes shown? (Your principal axes must pass through the chosen origin.) Note that the left two are point masses in the xy plane.





(A) Any axes.

(B) Axis through point; any axes $\perp$ to this.

(C) x, y, z axes.

(D) z axis; any axes in xy plane.

(E) z axis; axis through CM; axis $\perp$ to this.

Example 1: Point mass at the origin.

principal axes: any axes.

Example 2: Point mass at the point (x_0, y_0, z_0).

principal axes: axis through point, any axes perpendicular to this.

Example 3: Rectangle centered at the origin, as shown.

principal axes: z-axis, axes parallel to sides.

Example 4: Cylinder with axis as z-axis.

principal axes: z-axis, any axes in x-y plane.

Example 5: Square with one corner at origin, as shown.

principal axes: z axis, axis through CM, axis perp to this.

Let's first work through a freshman-physics-like collision problem that involves angular-momentum conservation. Then we'll work through a similar but trickier problem that requires us to project the motion onto the principal axes.

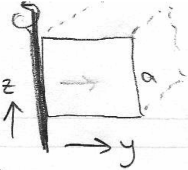
Taylor 10.16

(a) First show that the moment of inertia of a uniform cube of side a and mass M, rotating about an edge, is $(2/3)Ma^2$.

The cube is sliding with velocity $\boldsymbol{v}$ across a flat horizontal frictionless table when it hits a tiny step ($\perp \boldsymbol{v}$), and the leading lower edge comes abruptly to rest.

(b) Find the cube's angular velocity just after the collision.

(c) Find the minimum speed v for which the cube rolls over after hitting the step. (Actually just write down an equation for the minimum speed — the algebra is unenlightening.)



(a)

$$M = \rho a^3$$

$$I_{zz} = \int_0^a dx \int_0^a dy \int_0^a dz \; \rho \left(x^2+y^2\right)$$

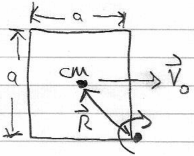
$$= \int_0^a dx \int_0^a dy \; a\rho(x^2+y^2) = a\rho \int_0^a dx \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{a}$$

$$= a\rho \int_0^a dx \left(x^2 a + \frac{1}{3}a^3\right) = a\rho \left[a\frac{x^3}{3} + \frac{1}{3}a^3 x\right]_{x=0}^{a}$$

$$= a\rho\left(\frac{a^4}{3} + \frac{a^4}{3}\right) = \frac{2}{3}\rho a^5 = \frac{2}{3}\left(\frac{M}{a^3}\right)\rho a^5$$

$$= \frac{2}{3} M a^2$$

(b)



$$L_i = \left|\vec{R}_{cm} \times M\vec{V}_0\right| = M\frac{a}{2}V_0$$

$$L_f = I\omega_f = \frac{2}{3}Ma^2\omega_f$$

$$\omega_f = \frac{\left(M\frac{a}{2}V_0\right)}{\frac{2}{3}Ma^2} = \frac{3V_0}{4a}$$

Angular momentum conservation w.r.t. chosen origin (at fixed position of step)

(c)

Energy conservation after collision

$$\frac{1}{2}I\omega_f^2 + Mg\,\underset{\text{(just after collision)}}{y_{cm}} = Mg\,\underset{\text{(when CM is above pivot)}}{y_{cm}}$$

$$\frac{1}{2}\left(\frac{2}{3}Ma^2\right)\left(\frac{M\frac{a}{2}V_0}{\frac{2}{3}Ma^2}\right)^2 + Mg\frac{a}{2} = Mg\frac{a}{\sqrt{2}}$$

after some unenlightening algebra,

$$(V_0)^2_{\text{minimum}} = ag\left(\frac{8}{3}\right)\left(\sqrt{2}-1\right) \simeq 1.1ag$$

Morin Exerise 9.38.

Fig. 9.56

9.38. Striking a triangle **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow? $\hat{z}$ out of page

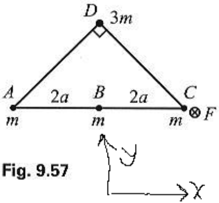


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

What is the post-impact motion of the CM?

What are the principal axes/moments?

Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.

Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\mathrm{CM}}$.

Morin Exerise 9.38.

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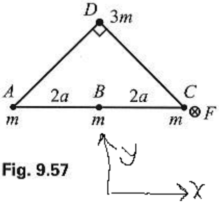


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

Halfway between B and D.

What is the post-impact motion of the CM?

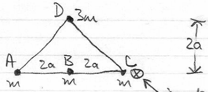
$\boldsymbol{V}_{\mathrm{cm}} = -P/(6m)\hat{\boldsymbol{z}}$ (and stays that way)

What are the principal axes/moments (w.r.t. CM)?

$\lambda_1 = 6ma^2$ ($\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$), $\lambda_2 = 8ma^2$ ($\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$), $\lambda_3 = 14ma^2$ ($\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$)

Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.

Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\mathrm{CM}}$.



$D = 3m$ @ $(0, +a, 0)$

put origin at CM

$A = m$ @ $(-2a, -a, 0)$

$B = m$ @ $(0, -a, 0)$

$C = m$ @ $(+2a, -a, 0)$

impulse at point $C = P = \int F\,dt$ into page

Immediately after impulsive blow $-P\hat{z}$ to C,

$$\vec{P}_{cm} = -P\hat{z} \Rightarrow \vec{V}_{cm} = -\frac{P}{6m}\hat{z}$$

$$\vec{L}_{\substack{\text{relative}\\ \text{to cm}}} = \vec{r} \times \vec{p} = (+2a, -a, 0) \times (0, 0, -P)$$

$$= (yP_z - zP_y,\ zP_x - xP_z,\ xP_y - yP_x)$$

$$= (aP, 2aP, 0) = aP(1, 2, 0)$$

Find principal moments:

confirm hunch that $I_{xy} = 0$: $I_{xy} = -\sum mxy = -ma^2(-2+2) = 0$

$$I_{xx} = \sum m(y^2+z^2) = \sum my^2 = ma^2(3+3) = 6ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = \sum mx^2 = ma^2(4+4) = 8ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = 14ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix} ma^2$$

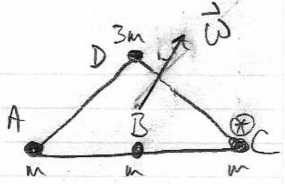
conveniently enough! So $\hat{x}, \hat{y}, \hat{z}$ are (initially) principal axes.

$$\vec{L} = (\lambda_1\omega_1,\ \lambda_2\omega_2,\ \lambda_3\omega_3)$$

$$\Rightarrow \omega_1 = \frac{aP}{6ma^2} \qquad \omega_2 = \frac{2aP}{8ma^2} \qquad \omega_3 = 0$$

$$\vec{\omega}_{\substack{\text{immediately}\\ \text{after impact}}} = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right)$$

$$\vec{V}_{\text{immediately after impact}} = \vec{V}_{cm} + \vec{\omega} \times \vec{r}$$



$$\vec{V}_{cm} = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_A = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (-2a, -a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{3}\right)$$

$$\vec{\omega} \times \vec{r}_B = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_C = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (+2a, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{2}{3}\right)$$

$$\vec{\omega} \times \vec{r}_D = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, +a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{6}\right)$$

$$\Rightarrow \quad \vec{V}_A = \frac{P}{m}\left(0, 0, \frac{1}{6}\right) \qquad \vec{V}_C = \frac{P}{m}\left(0, 0, -\frac{5}{6}\right)$$

$$\vec{V}_B = \frac{P}{m}\left(0, 0, -\frac{1}{3}\right) \qquad \vec{V}_D = \frac{P}{m}(0, 0, 0)$$

In[9]:=

```
{1 / 6, 1 / 4, 0} × {-2, -1, 0}
```

Out[9]= $\left\{0,\ 0,\ \frac{1}{3}\right\}$

In[10]:=

```
{1 / 6, 1 / 4, 0} × {0, -1, 0}
```

Out[10]=

$$\left\{0,\ 0,\ -\frac{1}{6}\right\}$$

In[11]:=

```
{1 / 6, 1 / 4, 0} × {+2, -1, 0}
```

Out[11]=

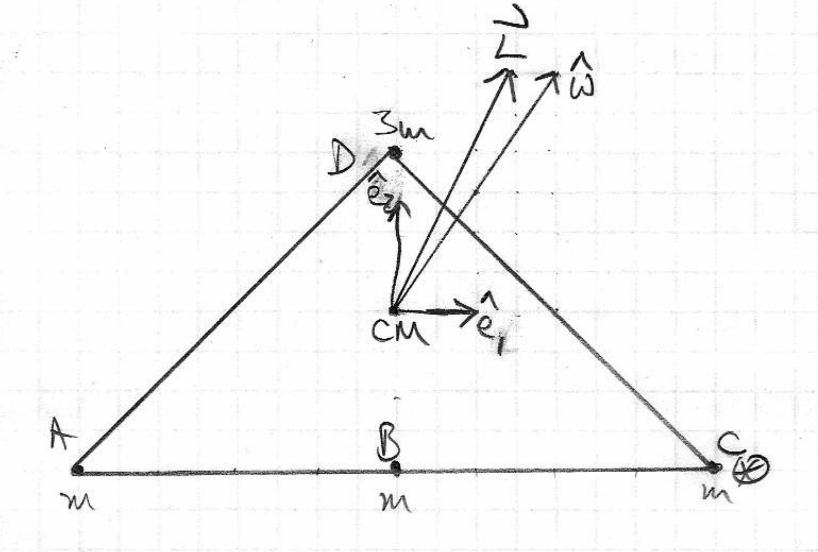
$$\left\{0,\ 0,\ -\frac{2}{3}\right\}$$

In[12]:=

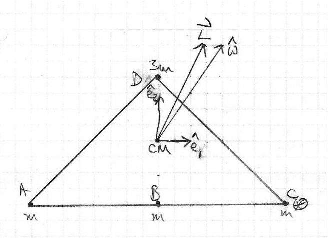
```
{1 / 6, 1 / 4, 0} × {0, +1, 0}
```

Out[12]=

$$\left\{0,\ 0,\ \frac{1}{6}\right\}$$



What will the subsequently happen to $\boldsymbol{V}_{\rm cm}$? To $\boldsymbol{L}$? To $\boldsymbol{\omega}$? To the orientations of the principal axes? With no applied torque, how does $\boldsymbol{\omega}$ evolve in time?



$\lambda_1 = 6ma^2,\ \lambda_2 = 8ma^2,\ \lambda_3 = 14ma^2.$

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1,2,0)$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3\omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Try other initial ω vectors:

https://www.youtube.com/watch?v=dVhGyxkBKzI
https://www.youtube.com/watch?v=4Ntgvun8GuY
https://www.youtube.com/watch?v=YKSEu_c3YdY

(For zero torque)

$$0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{space}\\ \text{axes}}} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{body}\\ \text{axes}}} + \vec{\omega} \times \vec{L}$$

$$(\dot{\omega}_1 \lambda_1, \dot{\omega}_2 \lambda_2, \dot{\omega}_3 \lambda_3) = -\vec{\omega} \times (\omega_1 \lambda_1, \omega_2 \lambda_2, \omega_3 \lambda_3)$$

$$= -(\omega_2 \omega_3 \lambda_3 - \omega_3 \omega_2 \lambda_2, \omega_3 \omega_1 \lambda_1 - \omega_1 \omega_3 \lambda_3, \omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1)$$

$$= (\omega_2 \omega_3 (\lambda_2 - \lambda_3), \omega_1 \omega_3 (\lambda_3 - \lambda_1), \omega_1 \omega_2 (\lambda_1 - \lambda_2))$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{(\lambda_2 - \lambda_3)}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

It's fun to consider e.g. $\lambda_3 > \lambda_2 > \lambda_1$ for tossed book.

$\lambda_3 > \lambda_2 > \lambda_1$. Start out e.g. about $\hat{e}_3$, with ω_1 and ω_2 both $\ll \omega_3$.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\ddot{\omega}_1 \simeq \dot{\omega}_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \underbrace{\omega_1 \omega_2}_{\text{"small} \times \text{small"}} \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$\Rightarrow \omega_3 \approx$ constant (varies very slowly)

$$\ddot{\omega}_1 = \omega_1 \left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)\left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right) = -\omega_1 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq \dot{\omega}_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = \omega_2 \left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right)\left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq -\omega_2 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\right) \equiv -\Omega^2 \omega_2$$

Start out about $\hat{e}_1$ $(\lambda_1 < \lambda_2 < \lambda_3)$, so ω_2 and ω_3 both $\ll \omega_1$ initially.

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \sim (\text{small})^2$$

$$\ddot{\omega}_2 \simeq \omega_1 \dot{\omega}_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = -\omega_2 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_2$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_3 \simeq \omega_1 \dot{\omega}_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = -\omega_3 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) \equiv -\Omega^2 \omega_3$$

Start out about $\hat{e}_2$, so ω_1 and $\omega_3 \ll \omega_2$ initially.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_2 = \omega_1\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \sim (\text{small})^2$$

$$\ddot{\omega}_1 \simeq \omega_2 \dot{\omega}_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \qquad \ddot{\omega}_3 \simeq \dot{\omega}_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_1 \simeq \omega_2 \left(\omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}\right) \frac{\lambda_2 - \lambda_3}{\lambda_1} = +\omega_1 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3}\right)$$

$$\ddot{\omega}_3 \simeq \left(\omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right) \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = +\omega_3 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3}\right)$$

$\Rightarrow$ exponential growth of ω_1, ω_3

$\longrightarrow$ initial motion about $\hat{e}_2$ won't stay about $\hat{e}_2$

Physics 351 — Wednesday, March 14, 2018

- Chapter 10 is (I think) the most difficult chapter in the book, so we will take our time to go through it slowly.
- HW7 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- On problem 1, you should first find $r(\phi)$ then work from the transformed radial equation (Taylor 8.41), where $u = 1/r$

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2\, u(\phi)^2}\, F$$