

Physics 351 — Monday, March 19, 2018

- ▶ Chapter 10 is (I think) the most difficult chapter in the book, so we will take our time to go through it slowly.
- ▶ Let's spend Wednesday and Friday this week working through some ch 7,8,9 problems together before the (3/26) midterm.
- ▶ FYI, you can see the midterm exams from the previous two times I taught this course at

http://positron.hep.upenn.edu/p351/files/midterm_2015.pdf

http://positron.hep.upenn.edu/p351/files/midterm_2017.pdf

Once you know how to calculate $\underline{\underline{I}}$, you can write the angular momentum

$$\mathbf{L} = \underline{\underline{I}} \boldsymbol{\omega}$$

and the kinetic energy

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \boldsymbol{\omega} \cdot (\underline{\underline{I}} \boldsymbol{\omega})$$

which generalize the freshman physics results

$$L = I\omega \qquad T = \frac{1}{2} I\omega^2$$

If we rotate coordinate axes into basis in which $\underline{\underline{I}}$ is diagonal, then

$$T = \frac{1}{2} (\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \lambda_3 \Omega_3^2) \qquad \mathbf{L} = (\lambda_1 \Omega_1, \lambda_2 \Omega_2, \lambda_3 \Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{I}}$ (i.e. are the 3 principal moments of inertia). So life is simpler in the “principal axes” basis.

If we rotate coordinate axes into basis in which $\underline{\underline{I}}$ is diagonal, then

$$T = \frac{1}{2} (\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \lambda_3 \Omega_3^2) \quad \mathbf{L} = (\lambda_1 \Omega_1, \lambda_2 \Omega_2, \lambda_3 \Omega_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underline{\underline{I}}$ (i.e. are the 3 principal moments of inertia).

Math fact: Given a real symmetric 3×3 matrix, $\underline{\underline{I}}$, there exist three orthonormal real vectors \mathbf{e}_i such that

$$\underline{\underline{I}} \mathbf{e}_i = \lambda_i \mathbf{e}_i$$

The unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (the eigenvectors of $\underline{\underline{I}}$) are called the principal axes of the rigid body. In most cases of interest, you can find the principal axes by symmetry, instead of having to solve the eigenvalue/eigenvector problem.

With $\underline{\underline{I}} = \int dm \begin{pmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{pmatrix}$, we get

$$\mathbf{L} = \underline{\underline{I}} \boldsymbol{\omega} \quad T = \frac{1}{2} \boldsymbol{\omega} \cdot (\underline{\underline{I}} \boldsymbol{\omega})$$

which generalize the familiar $L = I\omega$ and $T = \frac{1}{2} I\omega^2$.

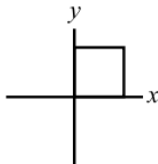
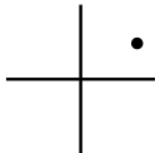
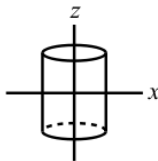
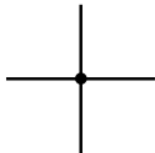
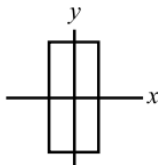
“Principal axes” basis simplifies these expressions considerably:

$$\underline{\underline{I}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = \frac{1}{2} (\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2) \quad \mathbf{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3)$$

So it's quite helpful to identify and use principal axes.

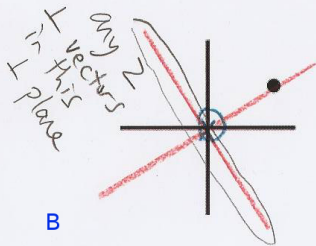
Using symmetry, what are the three principal axes of these five objects w.r.t. the origin of the coordinate axes shown? (Your principal axes must pass through the chosen origin.) Note that the left two are point masses in the xy plane.



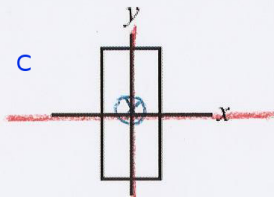
A



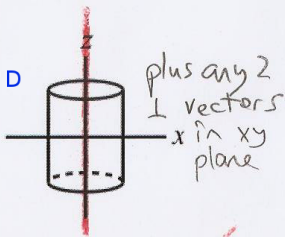
B



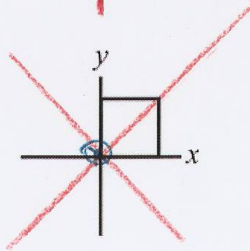
C



D



E



- (A) Any axes.
- (B) Axis through point; any axes \perp to this.
- (C) x, y, z axes.
- (D) z axis; any axes in xy plane.
- (E) z axis; axis through CM; axis \perp to this.

Example 1: Point mass at the origin.

principal axes: any axes.

Example 2: Point mass at the point (x_0, y_0, z_0) .

principal axes: axis through point, any axes perpendicular to this.

Example 3: Rectangle centered at the origin, as shown.

principal axes: z -axis, axes parallel to sides.

Example 4: Cylinder with axis as z -axis.

principal axes: z -axis, any axes in x - y plane.

Example 5: Square with one corner at origin, as shown.

principal axes: z axis, axis through CM, axis perp to this.

Let's first work through a freshman-physics-like collision problem that involves angular-momentum conservation. Then we'll work through a similar but trickier problem that requires us to project the motion onto the principal axes.

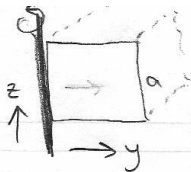
Taylor 10.16

(a) First show that the moment of inertia of a uniform cube of side a and mass M , rotating about an edge, is $(2/3)Ma^2$.

The cube is sliding with velocity \mathbf{v} across a flat horizontal frictionless table when it hits a tiny step ($\perp \mathbf{v}$), and the leading lower edge comes abruptly to rest.

(b) Find the cube's angular velocity just after the collision.

(c) Find the minimum speed v for which the cube rolls over after hitting the step. (Actually just write down an equation for the minimum speed — the algebra is unenlightening.)



①

$$M = \rho a^3$$

$$I_{zz} = \int_0^a dx \int_0^a dy \int_0^a dz \rho (x^2 + y^2)$$

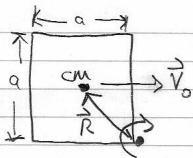
$$= \int_0^a dx \int_0^a dy a \rho (x^2 + y^2) = a \rho \int_0^a dx \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^a$$

$$= a \rho \int_0^a dx \left(x^2 a + \frac{1}{3} a^3 \right) = a \rho \left[a \frac{x^3}{3} + \frac{1}{3} a^3 x \right]_{x=0}^a$$

$$= a \rho \left(\frac{a^4}{3} + \frac{a^4}{3} \right) = \frac{2}{3} \rho a^5 = \frac{2}{3} \left(\frac{M}{a^3} \right) \rho a^5$$

$$= \frac{2}{3} M a^2$$

①



$$L_i = |\vec{R}_{cm} \times M \vec{V}_0| = M \frac{a}{2} v_0$$

$$L_f = I \omega_f = \frac{2}{3} M a^2 \omega_f$$

$$\omega_f = \frac{(M \frac{a}{2} v_0)}{\frac{2}{3} M a^2} = \frac{3 v_0}{4 a}$$

Angular momentum
conservation

w.r.t. chosen origin (at fixed position of step)

②

Energy conservation
after collision

$$\frac{1}{2} I \omega_f^2 + M g y_{cm} \quad = \quad M g y_{cm}$$

(just after collision) (when cm is above pivot)

$$\frac{1}{2} \left(\frac{2}{3} M a^2 \right) \left(\frac{M \frac{a}{2} v_0}{\frac{2}{3} M a^2} \right)^2 + M g \frac{a}{2} = M g \frac{a}{\sqrt{2}}$$

after some enlightening algebra,

$$(v_0)_{\text{minimum}}^2 = a g \left(\frac{8}{3} \right) (\sqrt{2} - 1) \approx 1.1 a g$$

Morin Exercise 9.38.

9.38. Striking a triangle **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m . Label them A, B, C, D , as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F dt = P$. What are the velocities of all the masses immediately after the blow?

Fig. 9.56

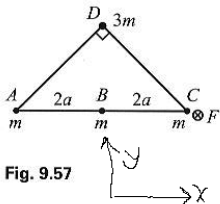


Fig. 9.57

2 out of page

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

What is the post-impact motion of the CM?

What are the principal axes/moments?

Write two different expressions for L , to find ω .

Use ω to find velocities w.r.t. CM, then combine with v_{CM} .

Morin Exercise 9.38.

9.38. Striking a triangle **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m . Label them A, B, C, D , as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F dt = P$. What are the velocities of all the masses immediately after the blow?

Fig. 9.56

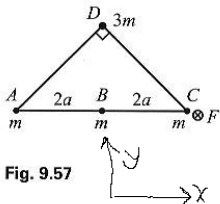


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

Halfway between B and D .

What is the post-impact motion of the CM?

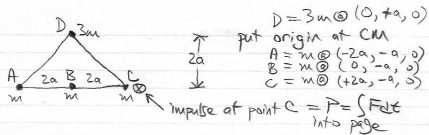
$\mathbf{V}_{\text{cm}} = -P/(6m)\hat{z}$ (and stays that way)

What are the principal axes/moments (w.r.t. CM)?

$\lambda_1 = 6ma^2$ ($\hat{e}_1 = \hat{x}$), $\lambda_2 = 8ma^2$ ($\hat{e}_2 = \hat{y}$), $\lambda_3 = 14ma^2$ ($\hat{e}_3 = \hat{z}$)

Write two different expressions for \mathbf{L} , to find $\boldsymbol{\omega}$.

Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with \mathbf{v}_{CM} .



Immediately after impulsive blow $-P\hat{z}$ to C,

$$\vec{P}_{cm} = -P\hat{z} \Rightarrow \vec{V}_{cm} = -\frac{P}{6m}\hat{z}$$

$$\begin{aligned}
 \vec{L}_{\text{relative to cm}} &= \vec{r} \times \vec{p} = (+2a, -a, 0) \times (0, 0, -P) \\
 &= (y p_z - z p_y, z p_x - x p_z, x p_y - y p_x) \\
 &= (aP, 2aP, 0) = aP(1, 2, 0)
 \end{aligned}$$

Find principal moments:

confirm hunch that $I_{xy} = 0$: $I_{xy} = -\sum m x y = -ma^2(-2+2) = 0$

$$I_{xx} = \sum m(y^2 + z^2) = \sum m y^2 = ma^2(3+3) = 6ma^2$$

$$I_{yy} = \sum m(x^2 + z^2) = \sum m x^2 = ma^2(4+4) = 8ma^2$$

$$I_{zz} = \sum m(x^2 + y^2) = 14ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix} ma^2$$

conveniently enough!
 so $\hat{x}, \hat{y}, \hat{z}$ are (initially)
 principal axes.

$$\vec{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3)$$

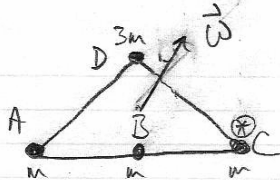
$$\Rightarrow \omega_1 = \frac{aP}{6ma^2} \quad \omega_2 = \frac{2aP}{8ma^2} \quad \omega_3 = 0$$

$$\vec{\omega}_{\text{immediately after impact}} = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right)$$

\vec{V} immediately
after
impact

$$= \vec{V}_{cm} + \vec{\omega} \times \vec{r}$$

$$\vec{V}_{cm} = \frac{P}{m} (0, 0, -\frac{1}{6})$$



$$\vec{\omega} \times \vec{r}_A = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right) \times (-2a, -a, 0) = \frac{P}{m} \left(0, 0, \frac{1}{3} \right)$$

$$\vec{\omega} \times \vec{r}_B = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right) \times (0, -a, 0) = \frac{P}{m} \left(0, 0, -\frac{1}{6} \right)$$

$$\vec{\omega} \times \vec{r}_C = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right) \times (+2a, -a, 0) = \frac{P}{m} \left(0, 0, -\frac{2}{3} \right)$$

$$\vec{\omega} \times \vec{r}_D = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right) \times (0, +a, 0) = \frac{P}{m} \left(0, 0, \frac{1}{6} \right)$$

$$\Rightarrow \left. \begin{aligned} \vec{V}_A &= \frac{P}{m} \left(0, 0, \frac{1}{6} \right) \\ \vec{V}_B &= \frac{P}{m} \left(0, 0, -\frac{1}{3} \right) \end{aligned} \right\} \begin{aligned} \vec{V}_C &= \frac{P}{m} \left(0, 0, -\frac{5}{6} \right) \\ \vec{V}_D &= \frac{P}{m} (0, 0, 0) \end{aligned}$$

In[9]:=

$$\{1/6, 1/4, 0\} \times \{-2, -1, 0\}$$

$$\text{Out[9]} = \left\{0, 0, \frac{1}{3}\right\}$$

$$\text{In[10]} := \{1/6, 1/4, 0\} \times \{0, -1, 0\}$$

Out[10]=

$$\left\{0, 0, -\frac{1}{6}\right\}$$

$$\text{In[11]} := \{1/6, 1/4, 0\} \times \{+2, -1, 0\}$$

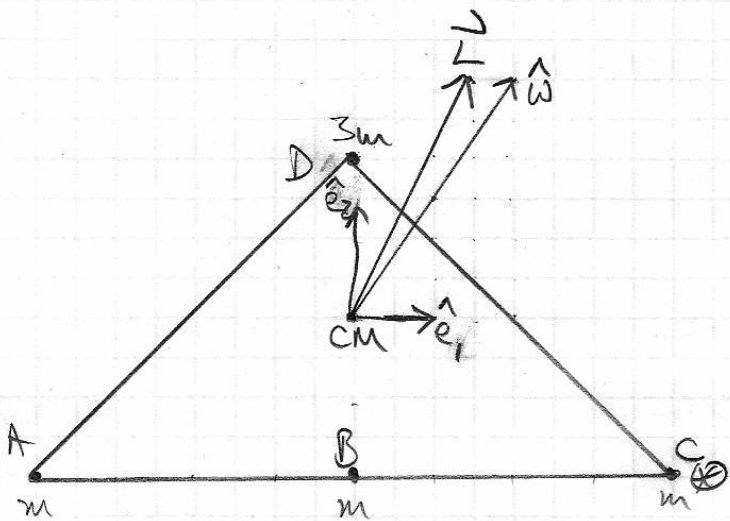
Out[11]=

$$\left\{0, 0, -\frac{2}{3}\right\}$$

$$\text{In[12]} := \{1/6, 1/4, 0\} \times \{0, +1, 0\}$$

Out[12]=

$$\left\{0, 0, \frac{1}{6}\right\}$$

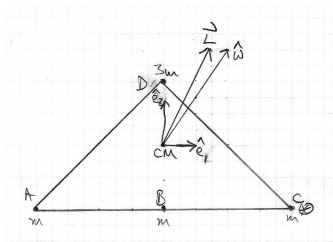


What will the subsequently happen to \mathbf{V}_{cm} ? To \mathbf{L} ? To $\boldsymbol{\omega}$? To the orientations of the principal axes? With no applied torque, how does $\boldsymbol{\omega}$ evolve in time?

$$\lambda_1 = 6ma^2, \lambda_2 = 8ma^2, \lambda_3 = 14ma^2.$$

Space and body axes coincide at $t = 0$.

$$\boldsymbol{\omega}_0 = \frac{P}{ma} \left(\frac{1}{6}, \frac{1}{4}, 0 \right). \quad \mathbf{L} \equiv aP(1, 2, 0).$$



$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3 \omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi

<https://www.youtube.com/watch?v=IMBRIyxDLss>

Try other initial $\boldsymbol{\omega}$ vectors:

<https://www.youtube.com/watch?v=dVhGyxkBKzI>

<https://www.youtube.com/watch?v=4Ntgvun8GuY>

https://www.youtube.com/watch?v=YKSEu_c3YdY

(For zero torque)

$$0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt} \right)_{\text{space axes}} = \left(\frac{d\vec{L}}{dt} \right)_{\text{body axes}} + \vec{\omega} \times \vec{L}$$

$$\begin{aligned} (\dot{\omega}_1 \lambda_1, \dot{\omega}_2 \lambda_2, \dot{\omega}_3 \lambda_3) &= - \vec{\omega} \times (\omega_1 \lambda_1, \omega_2 \lambda_2, \omega_3 \lambda_3) \\ &= - (\omega_2 \omega_3 \lambda_3 - \omega_3 \omega_2 \lambda_2, \omega_3 \omega_1 \lambda_1 - \omega_1 \omega_3 \lambda_3, \omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1) \\ &= (\omega_2 \omega_3 (\lambda_2 - \lambda_3), \omega_1 \omega_3 (\lambda_3 - \lambda_1), \omega_1 \omega_2 (\lambda_1 - \lambda_2)) \end{aligned}$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{(\lambda_2 - \lambda_3)}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

It's fun to consider e.g. $\lambda_3 > \lambda_2 > \lambda_1$ for tossed book.

$\lambda_3 > \lambda_2 > \lambda_1$. Start out e.g. about \hat{e}_3 ,
 with ω_1 and ω_2 both $\ll \omega_3$.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\ddot{\omega}_1 \approx \dot{\omega}_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \underbrace{\omega_1 \omega_2}_{\text{"Small x Small"}} \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$\Rightarrow \omega_3 \approx \text{constant}$ (varies very slowly)

$$\ddot{\omega}_1 = \omega_1 \left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \right) \left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) = -\omega_1 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2} \right)$$

$$\ddot{\omega}_2 \approx \dot{\omega}_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = \omega_2 \left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) \left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \right)$$

$$\ddot{\omega}_2 \approx -\omega_2 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2} \right) = -\Omega^2 \omega_2$$

Start out about \hat{e}_1 , ($\lambda_1 < \lambda_2 < \lambda_3$), so ω_2 and ω_3 both $\ll \omega_1$ initially.

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \quad \left| \quad \dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \sim (\text{small})^2 \right.$$

$$\ddot{\omega}_2 \approx \omega_1 \dot{\omega}_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = -\omega_2 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_2$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_3 \approx \omega_1 \dot{\omega}_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = -\omega_3 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_3$$

Start out about \hat{e}_2 , so ω_1 and $\omega_3 \ll \omega_2$ initially.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \quad \dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \sim (\text{small})^2$$

$$\ddot{\omega}_1 \simeq \omega_2 \dot{\omega}_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \quad \dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \quad \ddot{\omega}_3 \simeq \dot{\omega}_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_1 \simeq \omega_2 \left(\omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \right) \frac{\lambda_2 - \lambda_3}{\lambda_1} = +\omega_1 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

$$\ddot{\omega}_3 \simeq \left(\omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = +\omega_3 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

\Rightarrow exponential growth of ω_1, ω_3

\rightarrow initial motion about \hat{e}_2 won't stay about \hat{e}_2

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