Physics 351 — Wednesday, March 28, 2018

- I have graded 40/60 of the midterm exams. I'll finish tomorrow and will hand them back Friday.
- HW9 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- For the next 3–4 days, we will finally make our way through the rest of Chapter 10, which is (I think) the most difficult chapter in the book. The two significant topics after that are coupled oscillators and the Hamiltonian formalism, plus a bit of "enrichment" material.
- FYI — intuitive description of precession:

http://positron.hep.upenn.edu/p351/files/0331_george_abell_precession.pdf

Morin Exerise 9.38.

Fig. 9.56

9.38. **Striking a triangle** **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow? z out of page

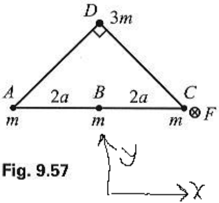


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.
What is the post-impact motion of the CM?
What are the principal axes/moments?
Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.
Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\mathrm{CM}}$.

Morin Exerise 9.38.

9.38. **Striking a triangle** **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow? z out of page

Fig. 9.56

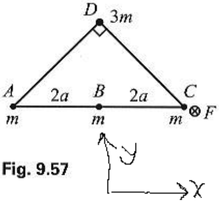


Fig. 9.57

Where is the CM? Let's call the CM (initially) $(0, 0, 0)$.

Halfway between B and D.

What is the post-impact motion of the CM?

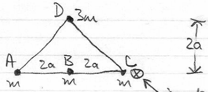
$\boldsymbol{V}_{\rm cm} = -P/(6m)\hat{\boldsymbol{z}}$ (and stays that way)

What are the principal axes/moments (w.r.t. CM)?

$\lambda_1 = 6ma^2$ ($\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$), $\lambda_2 = 8ma^2$ ($\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$), $\lambda_3 = 14ma^2$ ($\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$)

Write two different expressions for $\boldsymbol{L}$, to find $\boldsymbol{\omega}$.

Use $\boldsymbol{\omega}$ to find velocities w.r.t. CM, then combine with $\boldsymbol{v}_{\rm CM}$.



$D = 3m @ (0, +a, 0)$

put origin at CM

$A = m @ (-2a, -a, 0)$

$B = m @ (0, -a, 0)$

$C = m @ (+2a, -a, 0)$

impulse at point C $= P = \int F dt$ into page

Immediately after impulsive blow $-P\hat{z}$ to C,

$$\vec{P}_{cm} = -P\hat{z} \Rightarrow \vec{V}_{cm} = -\frac{P}{6m}\hat{z}$$

$$\vec{L}_{\substack{\text{relative}\\ \text{to cm}}} = \vec{r} \times \vec{p} = (+2a, -a, 0) \times (0, 0, -P)$$

$$= (yP_z - zP_y,\ zP_x - xP_z,\ xP_y - yP_x)$$

$$= (aP, 2aP, 0) = aP(1, 2, 0)$$

Find principal moments:

confirm hunch that $I_{xy} = 0$: $I_{xy} = -\sum mxy = -ma^2(-2+2) = 0$

$$I_{xx} = \sum m(y^2+z^2) = \sum my^2 = ma^2(3+3) = 6ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = \sum mx^2 = ma^2(4+4) = 8ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = 14ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix} ma^2$$

conveniently enough!

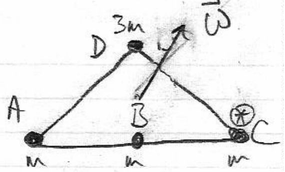
so $\hat{x}, \hat{y}, \hat{z}$ are (initially) principal axes.

$$\vec{L} = (\lambda_1\omega_1,\ \lambda_2\omega_2,\ \lambda_3\omega_3)$$

$$\Rightarrow \omega_1 = \frac{aP}{6ma^2} \qquad \omega_2 = \frac{2aP}{8ma^2} \qquad \omega_3 = 0$$

$$\vec{\omega}_{\substack{\text{immediately}\\ \text{after impact}}} = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right)$$

$$\vec{V}_{\substack{\text{immediately}\\\text{after}\\\text{impact}}} = \vec{V}_{cm} + \vec{\omega} \times \vec{r}$$



$$\vec{V}_{cm} = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_A = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (-2a, -a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{3}\right)$$

$$\vec{\omega} \times \vec{r}_B = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_C = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (+2a, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{2}{3}\right)$$

$$\vec{\omega} \times \vec{r}_D = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, +a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{6}\right)$$

$$\Rightarrow \quad \vec{V}_A = \frac{P}{m}\left(0, 0, \frac{1}{6}\right) \qquad \vec{V}_C = \frac{P}{m}\left(0, 0, -\frac{5}{6}\right)$$

$$\vec{V}_B = \frac{P}{m}\left(0, 0, -\frac{1}{3}\right) \qquad \vec{V}_D = \frac{P}{m}(0, 0, 0)$$

In[9]:=

```
{1 / 6, 1 / 4, 0} × {-2, -1, 0}
```

Out[9]= $\left\{0,\ 0,\ \frac{1}{3}\right\}$

In[10]:= `{1 / 6, 1 / 4, 0} × {0, -1, 0}`

Out[10]=

$\left\{0,\ 0,\ -\frac{1}{6}\right\}$

In[11]:= `{1 / 6, 1 / 4, 0} × {+2, -1, 0}`

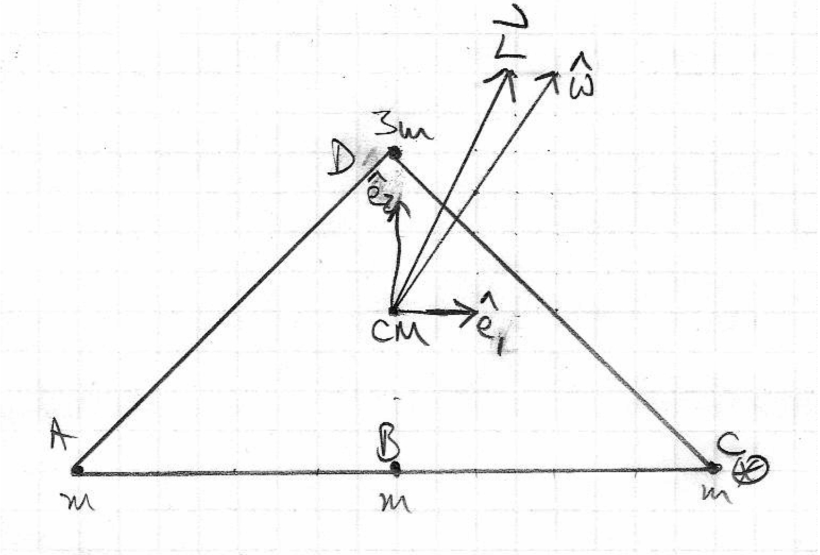
Out[11]=

$\left\{0,\ 0,\ -\frac{2}{3}\right\}$

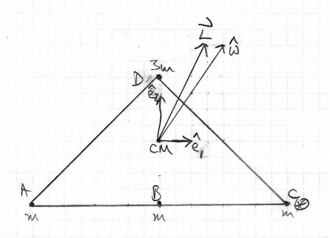
In[12]:= `{1 / 6, 1 / 4, 0} × {0, +1, 0}`

Out[12]=

$\left\{0,\ 0,\ \frac{1}{6}\right\}$



What will the subsequently happen to $\boldsymbol{V}_{\mathrm{cm}}$? To $\boldsymbol{L}$? To $\boldsymbol{\omega}$? To the orientations of the principal axes? With no applied torque, how does $\boldsymbol{\omega}$ evolve in time?



$\lambda_1 = 6ma^2,\ \lambda_2 = 8ma^2,\ \lambda_3 = 14ma^2.$

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1, 2, 0)$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3\omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Try other initial $\boldsymbol{\omega}$ vectors:

https://www.youtube.com/watch?v=dVhGyxkBKzI
https://www.youtube.com/watch?v=4Ntgvun8GuY
https://www.youtube.com/watch?v=YKSEu_c3YdY

(For zero torque)

$$0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{space}\\ \text{axes}}} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{body}\\ \text{axes}}} + \vec{\omega} \times \vec{L}$$

$$(\dot{\omega}_1\lambda_1, \dot{\omega}_2\lambda_2, \dot{\omega}_3\lambda_3) = -\vec{\omega} \times (\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3)$$

$$= -(\omega_2\omega_3\lambda_3 - \omega_3\omega_2\lambda_2, \omega_3\omega_1\lambda_1 - \omega_1\omega_3\lambda_3, \omega_1\omega_2\lambda_2 - \omega_2\omega_1\lambda_1)$$

$$= (\omega_2\omega_3(\lambda_2 - \lambda_3), \omega_1\omega_3(\lambda_3 - \lambda_1), \omega_1\omega_2(\lambda_1 - \lambda_2))$$

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{(\lambda_2 - \lambda_3)}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\dot{\omega}_2 = \omega_1\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

It's fun to consider e.g. $\lambda_3 > \lambda_2 > \lambda_1$ for tossed book.

$\lambda_3 > \lambda_2 > \lambda_1$. Start out e.g. about $\hat{e}_3$, with ω_1 and ω_2 both $\ll \omega_3$.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\ddot{\omega}_1 \simeq \dot{\omega}_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \underbrace{\omega_1 \omega_2}_{\text{"small x small"}} \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$\Rightarrow \omega_3 \approx$ constant $\left(\text{varies very slowly}\right)$

$$\ddot{\omega}_1 = \omega_1 \left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)\left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right) = -\omega_1 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq \dot{\omega}_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = \omega_2 \left(\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}\right)\left(\omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq -\omega_2 \left(\omega_3^2 \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\right) \equiv -\Omega^2 \omega_2$$

If you look down the $\hat{e}_3$ axis, you'll see the tip of $\boldsymbol{\omega}$ tracing out an ellipse whose ratio of axis lengths is $\sqrt{\frac{(\lambda_3-\lambda_2)\lambda_2}{(\lambda_3-\lambda_1)\lambda_1}}$.

$$\lambda_1\dot{\omega}_1 + \left[(\omega_1,\omega_2,\omega_3)\times(\lambda_1\omega_1,\lambda_2\omega_2,\lambda_3\omega_3)\right]_1 = 0$$

$$\lambda_1\dot{\omega}_1 + (\omega_2\lambda_3\omega_3 - \omega_3\lambda_2\omega_2) = 0$$

$$\lambda_1\dot{\omega}_1 + \omega_2\omega_3(\lambda_3-\lambda_2) = 0 \qquad (-\Omega^2\ )$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1} \rightarrow \ddot{\omega}_1 = \dot{\omega}_2\left(\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1}\right) = \omega_1\left(\omega_3^2\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\right)$$

$$\dot{\omega}_2 = \omega_3\omega_1\frac{\lambda_3-\lambda_1}{\lambda_2} \rightarrow \ddot{\omega}_2 = \dot{\omega}_1\left(\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2}\right) = \omega_2\left(\omega_3^2\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\right)$$

$$\omega_1 = A\cos\Omega t \rightarrow \dot{\omega}_1 = -\Omega A\sin\Omega t = -\frac{\Omega A}{B}\omega_2 \longrightarrow \omega_3\frac{\lambda_3-\lambda_2}{\lambda_1} = \frac{\Omega A}{B}$$

$$\omega_2 = B\sin\Omega t \rightarrow \dot{\omega}_2 = \Omega B\cos\Omega t = \frac{\Omega B}{A}\omega_1 \longrightarrow \omega_3\frac{\lambda_3-\lambda_1}{\lambda_2} = \frac{\Omega B}{A}$$

$$\frac{A^2}{B^2} = \frac{(\lambda_3-\lambda_2)\lambda_2}{(\lambda_3-\lambda_1)\lambda_1}$$

Start out about $\hat{e}_1$ $(\lambda_1 < \lambda_2 < \lambda_3)$, so ω_2 and ω_3 both $\ll \omega_1$ initially.

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \sim (\text{small})^2$$

$$\ddot{\omega}_2 \simeq \omega_1 \dot{\omega}_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = -\omega_2 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_2$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_3 \simeq \omega_1 \dot{\omega}_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = -\omega_3 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) \equiv -\Omega^2 \omega_3$$

Start out about $\hat{e}_2$, so ω_1 and $\omega_3 \ll \omega_2$ initially.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \boxed{\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \sim (\text{small})^2}$$

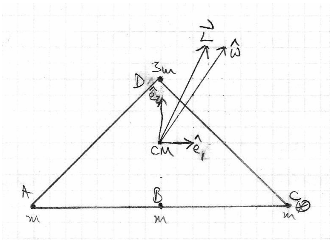
$$\ddot{\omega}_1 \simeq \omega_2 \dot{\omega}_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \qquad \ddot{\omega}_3 \simeq \dot{\omega}_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_1 \simeq \omega_2 \left(\omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \right) \frac{\lambda_2 - \lambda_3}{\lambda_1} = +\omega_1 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

$$\ddot{\omega}_3 \simeq \left(\omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = +\omega_3 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

$\Rightarrow$ exponential growth of ω_1, ω_3

$\rightarrow$ initial motion about $\hat{e}_2$ won't stay about $\hat{e}_2$



$\lambda_1 = 6ma^2$, $\lambda_2 = 8ma^2$, $\lambda_3 = 14ma^2$.

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1, 2, 0)$.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3 \omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Consider how you would go about calculating the (x, y, z) (space) positions of vertices A, C, D vs. time. I did it by keeping track of the (x, y, z) coordinates of the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ as a function of time.

(Neglecting $\vec{v}_{cm}$, as I did in animation)

$\vec{r}_A = -2a\hat{e}_1 - a\hat{e}_2 \qquad \vec{r}_C = +2a\hat{e}_1 - a\hat{e}_2$

$\vec{r}_D = +a\hat{e}_2$

Initially, $\hat{e}_1 = (1,0,0) \quad \hat{e}_2 = (0,1,0) \quad \hat{e}_3 = (0,0,1)$

We are given initial $\underset{\sim}{\omega}_{body}$.

$\underset{\sim}{\omega}_{space} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$

$\left(\frac{d\hat{e}_1}{dt}\right)_{space} = \underset{\sim}{\omega}_{space} \times \hat{e}_1 \quad \ldots\ldots \quad \left(\frac{d\hat{e}_3}{dt}\right)_{space} = \underset{\sim}{\omega}_{space} \times \hat{e}_3$

Lets me update $\hat{e}_1, \hat{e}_2, \hat{e}_3$ in x, y, z (space) coordinates

Then $\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$ etc.

lets me update $\underset{\sim}{\omega}_{body}$. Iterate for each frame of animation.

$$\underset{\sim}{\Gamma} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{space} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (\lambda_1 \dot{\omega}_1, \lambda_2 \dot{\omega}_2, \lambda_3 \dot{\omega}_3) + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + (\omega_1 L_2 - \omega_2 L_1)$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + \omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1 \omega_2$$

$$\lambda_3 \dot{\omega}_3 = \Gamma_3 + (\lambda_1 - \lambda_2)\omega_1 \omega_2$$

$$\lambda_1 \dot{\omega}_1 = \Gamma_1 + (\lambda_2 - \lambda_3)\omega_2 \omega_3$$

$$\lambda_2 \dot{\omega}_2 = \Gamma_2 + (\lambda_3 - \lambda_1)\omega_3 \omega_1$$

Torque-free precession of symmetric top (more on this later):

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\lambda_3 \dot{\omega}_3 = \Gamma_3 \quad (I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 = \text{const.}$ Suppose $\underset{\sim}{\Gamma} = 0$.

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\,\omega_2 \omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\,\omega_3 \omega_1 \end{aligned}\right\} \text{let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda}\,\omega_3$$

$$\dot{\omega}_1 = \Omega\,\omega_2 \;, \quad \dot{\omega}_2 = -\Omega\,\omega_1$$

$$\rightarrow \quad \begin{aligned} \omega_1 &= \omega_0 \cos \Omega t \\ \omega_2 &= -\omega_0 \sin \Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{\omega}$ precesses about $\hat{e}_3$ with frequency Ω.
As seen from the body frame, what does $\boldsymbol{L}$ do?

What does the situation look like from the space frame?

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\lambda_3 \dot{\omega}_3 = \Gamma_3 \quad (I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 = \text{const.}$ Suppose $\underset{\sim}{\Gamma} = 0$.

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\omega_2 \omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\omega_3 \omega_1 \end{aligned}\right\} \text{let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda} \omega_3$$

$$\dot{\omega}_1 = \Omega \omega_2 \ , \quad \dot{\omega}_2 = -\Omega \omega_1$$

$$\begin{aligned} \rightarrow \quad \omega_1 &= \omega_0 \cos \Omega t \\ \omega_2 &= -\omega_0 \sin \Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ precess about (fixed) $\hat{\boldsymbol{e}}_3$ with frequency $\Omega_b \equiv \Omega = \omega_3(\lambda - \lambda_3)/\lambda$, where $\lambda = \lambda_1 = \lambda_2$.

As seen from the space frame, $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{\omega}$ precess about (fixed) $\boldsymbol{L}$, at a frequency that takes some effort to calculate. (You'll calculate the space-frame precession frequency, Ω_s, on a HW problem next week. It is much more involved than you might expect.)

Video from two 2015 students traveling back from spring break:
https://www.youtube.com/watch?v=bVpPp1e_lZ4

Astronaut version:
https://youtu.be/fPI-rSwAQNg

Cosmonaut version (!): Dzhanibekov effect
https://youtu.be/dL6Pt1O_gSE

https://www.youtube.com/watch?v=BGRWg4aV2mw

Someone's quasi-intuitive explanation:
http://mathoverflow.net/questions/81960/
the-dzhanibekov-effect-an-exercise-in-mechanics-or-fiction-explain-mathemat

(Taylor 10.35) A rigid body consists of:

$$\begin{aligned} m \text{ at } (a,0,0) &= a(1,\ 0,\ \ 0) \\ 2m \text{ at } (0,a,a) &= a(0,\ 1,\ \ 1) \\ 3m \text{ at } (0,a,-a) &= a(0,\ 1,-1) \end{aligned}$$

Find inertia tensor $\underline{\underline{\boldsymbol{I}}}$, its principal moments, and the principal axes.

$$I_{xx} = \sum m(y^2+z^2) = ma^2(2\cdot 2+2\cdot 3) = 10ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{xy} = -\sum mxy = -ma^2(0) = 0$$

$$I_{xz} = -\sum mxz = -ma^2(0) = 0$$

$$I_{yz} = -\sum myz = -ma^2(2-3) = ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} ma^2$$

$$\underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda\,\underset{\sim}{\omega} \Rightarrow \left(\underline{\underline{I}} - \lambda \mathbb{1}\right)\underset{\sim}{\omega} = 0$$

$$\Rightarrow \det\left(\underline{\underline{I}} - \lambda \mathbb{1}\right) = 0$$

$$0 = (10-\lambda)(6-\lambda)^2 - (10-\lambda) \Rightarrow \lambda = 10 \text{ or } (6-\lambda)^2 = 1$$

$$6-\lambda = 1 \Rightarrow \lambda = 5, \quad 6-\lambda = -1 \Rightarrow \lambda = 7 \qquad \lambda \in \{10, 7, 5\}$$

$$\lambda \in \{10, 7, 5\}\, ma^2$$

$$0 = \begin{pmatrix} 10-10 & 0 & 0 \\ 0 & 6-10 & 1 \\ 0 & 1 & 6-10 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4y+z \\ y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = z_1 = 0 \Rightarrow \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-7 & 0 & 0 \\ 0 & 6-7 & 1 \\ 0 & 1 & 6-7 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3x \\ z-y \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0,\ y_2 = z_2 \Rightarrow \hat{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-5 & 0 & 0 \\ 0 & 6-5 & 1 \\ 0 & 1 & 6-5 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5x \\ y+z \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3 = 0,\ y_3 = -z_3 \Rightarrow \hat{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

eigenvectors {{10,0,0},{0,6,1},{0,1,6}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}\right]$$

Results:

$v_1 = (1, 0, 0)$

$v_2 = (0, 1, 1)$

$v_3 = (0, -1, 1)$

Corresponding eigenvalues:

$\lambda_1 = 10$

$\lambda_2 = 7$

$\lambda_3 = 5$

```
In[1]:= m = {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

Out[1]= {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

In[2]:= MatrixForm[m]

2]//MatrixForm=
```

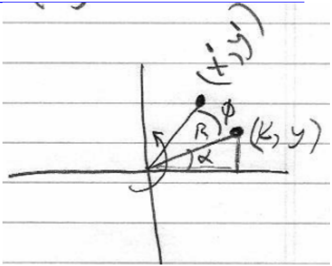
$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

```
In[3]:= Eigenvalues[m]

Out[3]= {10, 7, 5}

In[4]:= Eigenvectors[m]

Out[4]= {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}

In[5]:= Eigensystem[m]

Out[5]= {{10, 7, 5},
          {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}}
```

One useful tool for relating the fixed $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ axes to the rigid body's $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$, axes is the "Euler angles," ϕ, θ, ψ.

(Another way, which I used in the simulation program for the struck triangle, is simply to keep track instant-by-instant of the x, y, z components of $\hat{\boldsymbol{e}}_1(t)$, $\hat{\boldsymbol{e}}_2(t)$, $\hat{\boldsymbol{e}}_3(t)$. But if you're given the three Euler angles, you can compute the x, y, z components of the body axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$.)

Question: Suppose I rotate the vector $(x, y) = R(\cos\alpha, \sin\alpha)$ by an angle ϕ (about the origin). How would you write x' as a linear combination of x and y? How about y' as a linear combination of x and y?

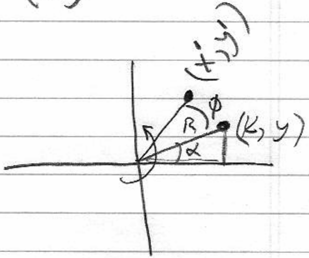


Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = x\cos\phi - y\sin\phi$$

$$y' = x\sin\phi + y\cos\phi$$



$$(x, y) = R(\cos\alpha, \sin\alpha)$$

$$(x', y') = R\left(\cos(\alpha+\phi), \sin(\alpha+\phi)\right)$$

$$= R\left(\cos\alpha\cos\phi - \sin\alpha\sin\phi, \ \sin\alpha\cos\phi + \cos\alpha\sin\phi\right)$$

$$= \left(x\cos\phi - y\sin\phi, \ y\cos\phi + x\sin\phi\right)$$

Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

x', y' as above ; $z' = z$

Rotate by angle θ about $\hat{y}'$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

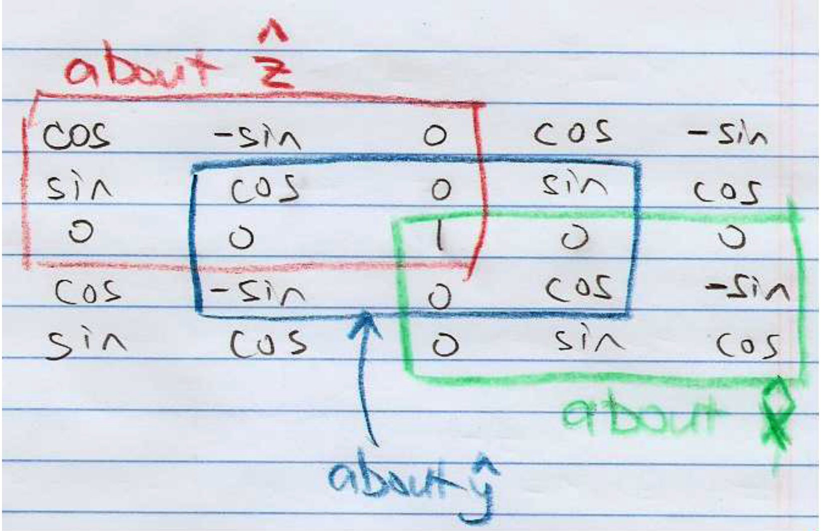
Mnemonic: for infinitessimal rotation angle $\epsilon \ll 1$, $\boldsymbol{r} \to \boldsymbol{r} + \epsilon\hat{\boldsymbol{\omega}} \times \boldsymbol{r}$. So for rotation about $\hat{\boldsymbol{y}}$, $(1,0,0) \to (1,0,-\epsilon)$, since $\epsilon\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\epsilon\hat{\boldsymbol{z}}$.

The hardest part of writing down 3×3 rotation matrices is remembering where to put the minus sign.

$$\begin{matrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{matrix}$$

Once you've worked out one case correctly (e.g. from a diagram), here's a trick (thanks to 2015 student Adam Zachar) for working out the other two ...

Just add two more columns and two more rows, following the cycles: xyz, yzx, zxy. Then draw boxes of size 3×3.



(Check previous result using Mathematica.)

```
In[1]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & -\mathrm{Sin}[\phi] & 0 \\ \mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[2]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] \\ 0 & 1 & 0 \\ -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix}$$

```
In[3]:= RotationMatrix[α, {1, 0, 0}] // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] \\ 0 & \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] \end{pmatrix}$$

Euler angles: can move (x, y, z) axes to arbitrary orientation.

Rotate by ϕ about $\hat{z}$

Then rotate by θ about $\hat{y}'$ $(\hat{e}_2')$

Then rotate by ψ about $\hat{z}''$ $(\hat{e}_3)$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \left(\begin{array}{c|c|c} c\theta c\phi c\psi - s\phi s\psi & -c\theta s\phi c\psi - c\phi s\psi & s\theta c\psi \\ c\theta c\phi s\psi + s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\psi \\ -s\theta c\phi & s\theta s\phi & c\theta \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In[2]:= `RotationMatrix[ϕ] // MatrixForm`

Out[2]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & -\mathrm{Sin}[\phi] \\ \mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] \end{pmatrix}$$

In[4]:= `RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm`

Out[4]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & -\mathrm{Sin}[\phi] & 0 \\ \mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[5]:= `RotationMatrix[θ, {0, 1, 0}] // MatrixForm`

Out[5]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] \\ 0 & 1 & 0 \\ -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix}$$

In[10]:=
```
r1 = RotationMatrix[ϕ, {0, 0, 1}];
r2 = RotationMatrix[θ, {0, 1, 0}];
r3 = RotationMatrix[ψ, {0, 0, 1}];
r3 . r2 . r1 // MatrixForm
```

Out[13]//MatrixForm=

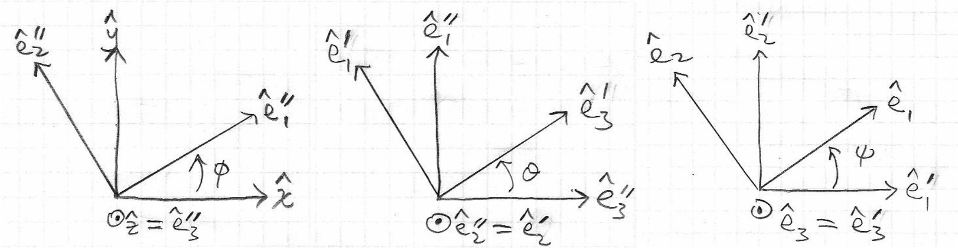
$$\begin{pmatrix} \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] \\ \mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] \\ -\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] & \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] & \mathrm{Cos}[\theta] \end{pmatrix}$$

Suppose the Euler angles ϕ, θ, ψ vary with time, as the body orientation changes w.r.t. the space frame's fixed x, y, z axes.

I'll write out more steps than Taylor does, and I may confuse you by saying $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \to (\hat{\boldsymbol{e}}_1'', \hat{\boldsymbol{e}}_2'', \hat{\boldsymbol{e}}_3'') \to (\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3') \to (\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$.

I do this so that my $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$ are the same as Taylor's.

1. Rotate by ϕ about $\hat{\boldsymbol{z}} \to \hat{\boldsymbol{e}}_1''$, $\hat{\boldsymbol{e}}_2''$. ($\hat{\boldsymbol{e}}_3'' = \hat{\boldsymbol{z}}$.)
2. Rotate by θ about $\hat{\boldsymbol{e}}_2'' \to \hat{\boldsymbol{e}}_1'$, $\hat{\boldsymbol{e}}_3'$. ($\hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_2''$.)
3. Rotate by ψ about $\hat{\boldsymbol{e}}_3' \to \hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$. ($\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_3'$.)



$$\boldsymbol{\omega} \;=\; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2'' + \dot{\psi}\,\hat{\boldsymbol{e}}_3' \;=\; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$$

Next, project $\boldsymbol{\omega}$ onto more convenient sets of unit vectors.

(orthogonal matrix: inverse = transpose)

$$\left.\begin{aligned} \hat{e}_1'' &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{e}_2'' &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{e}_3'' &= \hat{z} \end{aligned}\right\} \qquad \begin{aligned} \hat{x} &= \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi \\ \hat{y} &= \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1' &= \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \\ \hat{e}_3' &= \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \\ \hat{e}_2' &= \hat{e}_2'' \end{aligned}\right\} \qquad \begin{aligned} \hat{e}_1'' &= \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta \\ \hat{e}_3'' &= -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1 &= \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \\ \hat{e}_2 &= -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \\ \hat{e}_3 &= \hat{e}_3' \end{aligned}\right\} \qquad \begin{aligned} \hat{e}_1' &= \hat{e}_1\cos\psi - \hat{e}_2\sin\psi \\ \hat{e}_2' &= \hat{e}_1\sin\psi + \hat{e}_2\cos\psi \end{aligned}$$

$$\underset{\sim}{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' = \dot{\phi}\hat{z} + \dot{\theta}(-\hat{x}\sin\phi + \hat{y}\cos\phi) + \dot{\psi}(\hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi,\ +\dot{\theta}\cos\phi,\ \dot{\phi}) + \dot{\psi}\left(\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \cos\theta(\hat{z})\right)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi,\ \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,\ \dot{\phi} + \dot{\psi}\cos\theta)$$

(IN SPACE AXES)

$$\left.\begin{aligned} \hat{e}_1'' &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{e}_2'' &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{e}_3'' &= \hat{z} \end{aligned}\right\} \quad \begin{aligned} \hat{x} &= \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi \\ \hat{y} &= \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1' &= \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \\ \hat{e}_3' &= \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \\ \hat{e}_2' &= \hat{e}_2'' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1'' &= \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta \\ \hat{e}_3'' &= -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1 &= \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \\ \hat{e}_2 &= -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \\ \hat{e}_3 &= \hat{e}_3' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1' &= \hat{e}_1\cos\psi - \hat{e}_2\sin\psi \\ \hat{e}_2' &= \hat{e}_1\sin\psi + \hat{e}_2\cos\psi \end{aligned}$$

In the "space" basis [proof on previous page]:

$$\boldsymbol{\omega} \;=\; (-\dot\theta\sin\phi + \dot\psi\sin\theta\cos\phi)\hat{\boldsymbol{x}} + (\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (\dot\phi + \dot\psi\cos\theta)\hat{\boldsymbol{z}}$$

In the "body" basis [proof on next page]:

$$\boldsymbol{\omega} \;=\; (-\dot\phi\sin\theta\cos\psi + \dot\theta\sin\psi)\hat{\boldsymbol{e}}_1 + (\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\hat{\boldsymbol{e}}_2 + (\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}_3$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$): in the "primed" basis (i.e. before the final rotation by ψ about $\hat{\boldsymbol{e}}_3$). Note that $\hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_3$.

$$\boldsymbol{\omega} \;=\; (-\dot\phi\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\dot\theta)\hat{\boldsymbol{e}}_2' \;+\; (\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}_3'$$

This last one is easiest to see if you consider the instant at which $\psi = 0$.

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\begin{aligned}
\underset{\sim}{\omega} &= \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\hat{e}_3'' + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\left(-\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta\right) + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1'\right) + \dot{\phi}\cos\theta\left(\hat{e}_3'\right)\right] + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1\cos\psi - \hat{e}_2\sin\psi\right) + \dot{\phi}\cos\theta\,\hat{e}_3\right] + \dot{\theta}\left(\hat{e}_1\sin\psi + \hat{e}_2\cos\psi\right) + \dot{\psi}\hat{e}_3' \\
&= \hat{e}_1\left(-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right) + \hat{e}_2\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + \hat{e}_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)
\end{aligned}$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$):

$$\boldsymbol{\omega} \;=\; (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}'_1 \,+\, (\dot{\theta})\hat{\boldsymbol{e}}'_2 \,+\, (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}'_3$$

Now, how do we write the top's angular momentum $\boldsymbol{L}$ and kinetic energy T? How about the Lagrangian?

Most convenient for symmetric top ($\lambda_1 = \lambda_2$):

$$\boldsymbol{\omega} \;=\; (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

Now, how do we write the top's angular momentum $\boldsymbol{L}$ and kinetic energy T? How about the Lagrangian?

$$\boldsymbol{L} \;=\; (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

Which coordinates (Euler angles) are ignorable? What are the corresponding conserved momenta?

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' + (\dot{\theta})\hat{\boldsymbol{e}}_2' + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$\boldsymbol{L} = (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' + (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}_2' + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)} \equiv P_\psi \equiv \text{const.} = \lambda_3\omega_3 = \boxed{L_3}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta \equiv P_\phi \equiv \text{const.}$$

$$= \boxed{(L_z - L_3\cos\theta)} + \boxed{L_3}\cos\theta = L_z$$

Digression: $\hat{z} = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$

$$L_z = \underset{\sim}{L}\cdot\hat{z} = \lambda_1\dot{\phi}\sin^2\theta + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta$$

$$L_z = \lambda_1\dot{\phi}\sin^2\theta + L_3\cos\theta$$

$$\Rightarrow \lambda_1\dot{\phi}\sin^2\theta = L_z - L_3\cos\theta$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

θ equation of motion: (derivation 2 pages ahead)

$$\lambda_1\ddot\theta = \lambda_1\boxed{\dot\phi^2}\sin\theta\cos\theta - (\lambda_3\omega_3)\boxed{\dot\phi}\sin\theta + MgR\sin\theta$$

First consider case where θ = constant. Since $\lambda_1\dot\phi\sin^2\theta = L_z - L_3\cos\theta$, $\theta = \text{const} \Rightarrow \dot\phi = \text{const} \equiv \Omega$

$$0 = \lambda_1\boxed{\Omega^2}\cancel{\sin\theta}\cos\theta - \lambda_3\omega_3\boxed{\Omega}\cancel{\sin\theta} + MgR\cancel{\sin\theta}$$

$$(\lambda_1\cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR = 0$$

$$\Omega = \frac{\lambda_3\omega_3 \pm\sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1\cos\theta MgR}}{2\lambda_1\cos\theta}$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm\sqrt{1 - \frac{4\lambda_1\cos\theta MgR}{(\lambda_3\omega_3)^2}}\right)$$

$$\Omega = \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1 \cos\theta M g R}{(\lambda_3 \omega_3)^2}}\right)$$

has 2 real solutions if $(\lambda_3 \omega_3)^2 > 4\lambda_1 M g R \cos\theta$
("if ω_3 is large enough")

Math is simplest if $(\lambda_3 \omega_3)^2 \gg 4\lambda_1 M g R \cos\theta$
("ω_3 is very large")

$$\Omega = \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta}\left(1 \pm \left(1 - \frac{2\lambda_1 M g R \cos\theta}{(\lambda_3 \omega_3)^2}\right)\right)$$

$$\boxed{\Omega_+ \simeq \frac{\lambda_3 \omega_3}{\lambda_1 \cos\theta}} \qquad \Omega_- \simeq \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta} \cdot \frac{2\lambda_1 M g R \cos\theta}{(\lambda_3 \omega_3)^2} = \boxed{\frac{MgR}{\lambda_3 \omega_3} = \Omega_-}$$

This is Ω_{space} for tossed football with no torque

(See HW)

This can be seen from

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$MgR\sin\theta = \lambda_3 \omega_3 \, \Omega \sin\theta$$

(Skip: Just in case you wanted to see the θ EOM derived.)

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})(-\dot{\phi}\sin\theta) + MgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\lambda_1\dot{\theta}) = \lambda_1\ddot{\theta}$$

$$\lambda_1\ddot{\theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta - \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}\sin\theta + MgR\sin\theta$$

$$\lambda_1\ddot{\theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta - \lambda_3\omega_3\dot{\phi}\sin\theta + MgR\sin\theta$$

$$\lambda_1 \dot\phi \sin^2\theta = L_z - L_3\cos\theta \Rightarrow \dot\phi = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$$

$$E = T + U = \tfrac{1}{2}\lambda_1\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \tfrac{1}{2}\lambda_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 + MgR\cos\theta$$

$$E = \tfrac{1}{2}\lambda_1\sin^2\theta\,\dot\phi^2 + \tfrac{1}{2}\lambda_1\dot\theta^2 + \tfrac{1}{2}\lambda_3 \omega_3^2 + MgR\cos\theta$$

$$E = \frac{\lambda_1\sin^2\theta}{2}\cdot\frac{(L_z - L_3\cos\theta)^2}{\lambda_1^2\sin^4\theta} + \frac{\lambda_1\dot\theta^2}{2} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

$$E = \frac{\lambda_1\dot\theta^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

$$E = \tfrac{1}{2}\lambda_1\dot\theta^2 + U_{eff}(\theta)$$

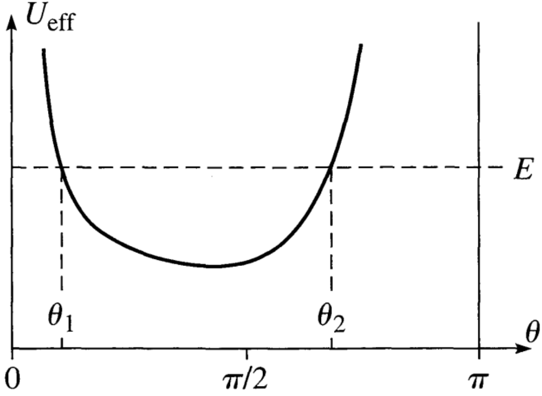
(one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2

$$E = \frac{\lambda_1 \dot{\theta}^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

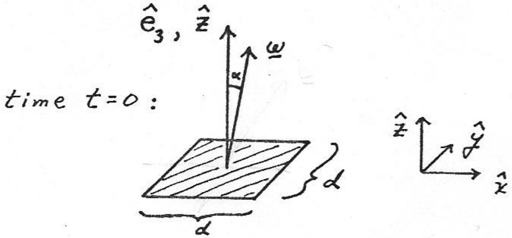
$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta) \qquad \text{(one dimensional problem)}$$

θ "nutates" back and forth between θ_1 and θ_2



From the final exam for the course I took, fall 1990. (This turns out to be the same problem as appears in Feynman's story of the cafeteria plate that wobbles as it flies through the air.)

An infinitely thin, uniform, square plate of mass m and side d is allowed to undergo rotation. At time $t = 0$, the normal to the plate, $\hat{e}_3$, is aligned with $\hat{z}$, but the angular velocity vector $\underline{\omega}$ deviates from $\hat{z}$ by a small angle α. Work the entire problem to first order in α, i.e. drop terms of $O(\alpha^2)$ or higher.



(a) Show $I = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find I_0.

(b) Find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate.

(c) When is this maximum deviation first reached?

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\quad \lambda_3 \dot{\omega}_3 = \Gamma_3 \quad (I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 =$ const. Suppose $\underset{\sim}{\Gamma} = 0$.

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\omega_2\omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\omega_3\omega_1 \end{aligned}\right\} \text{let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda}\omega_3$$

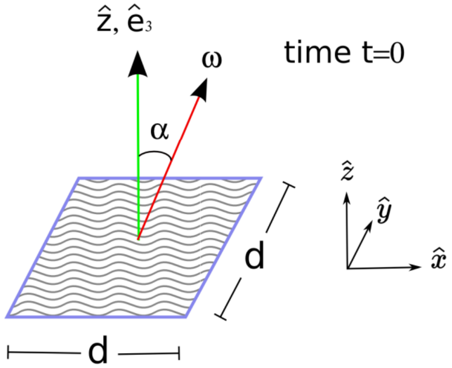
$$\dot{\omega}_1 = \Omega\,\omega_2 \,, \quad \dot{\omega}_2 = -\Omega\,\omega_1$$

$$\longrightarrow \quad \begin{aligned} \omega_1 &= \omega_0 \cos \Omega t \\ \omega_2 &= -\omega_0 \sin \Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ precess about (fixed) $\hat{\boldsymbol{e}}_3$ with frequency $\Omega_b \equiv \Omega = \omega_3(\lambda - \lambda_3)/\lambda$, where $\lambda = \lambda_1 = \lambda_2$.

As seen from the space frame, $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{\omega}$ precess about (fixed) $\boldsymbol{L}$, at frequency $\Omega_s = L/\lambda_1$, which you'll prove in the HW.

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



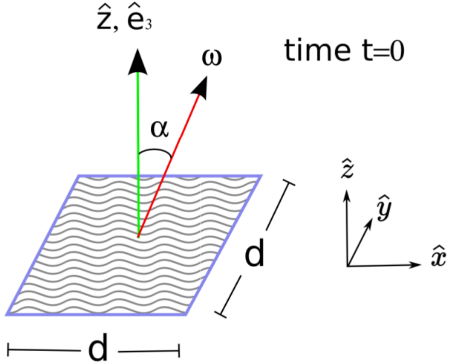
(a) Show that $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0.

(b) Calculate $\boldsymbol{L}$ at $t = 0$.

(c) Sketch $\hat{e}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$.

(d) Draw/label "body cone" and "space cone" on your sketch.

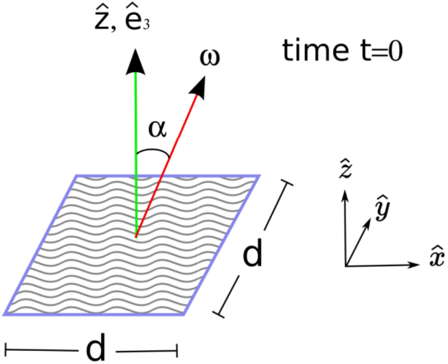
8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(e) Calculate precession frequencies Ω_{body} and Ω_{space}. Indicate directions of precession vectors $\boldsymbol{\Omega}_{\textbf{body}}$ and $\boldsymbol{\Omega}_{\textbf{space}}$ on drawing.

(f) You argue in HW that $\boldsymbol{\Omega}_{\textbf{space}} = \boldsymbol{\Omega}_{\textbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\textbf{space}}$ and $\boldsymbol{\Omega}_{\textbf{body}}$ that you calculate for $t = 0$.

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(g) Find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{e}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α.

(h) When is this maximum deviation first reached?

video: `https://www.youtube.com/watch?v=oH-dlrIF010`

Physics 351 — Wednesday, March 28, 2018

- I have graded 40/60 of the midterm exams. I'll finish tomorrow and will hand them back Friday.
- HW9 due Friday. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- For the next 3–4 days, we will finally make our way through the rest of Chapter 10, which is (I think) the most difficult chapter in the book. The two significant topics after that are coupled oscillators and the Hamiltonian formalism, plus a bit of "enrichment" material.
- FYI — intuitive description of precession:

http://positron.hep.upenn.edu/p351/files/0331_george_abell_precession.pdf