Physics 351 — Wednesday, April 4, 2018

- Pick up **worksheet (!)** on your way in.
- HW10 due this Friday. I tried (!) to make it short. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- What if in future years I did weekly quizzes (simple repeat of a HW problem, after HW has been graded & handed back with solutions), but scaled them `scaled = min(raw/0.66,1.00)` so that earning 2/3 of semester quiz points earns you 100%?
- Let's start today by trying to restate more clearly Monday's discussion of $\mathbf{\Omega}_{\text{space}}$ and $\mathbf{\Omega}_{\text{body}}$. The idea is to understand the torque-free wobbling motion of a tossed object that has axial symmetry: oblate (frisbee-like) or prolate (US-football-like). That will segue into the worksheet, which resembles HW q8.

If $\tilde{\Gamma} = 0$ then

$$0 = \left(\frac{dL}{dt}\right)_{space} = \left(\frac{dL}{dt}\right)_{body} + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\Rightarrow (\lambda_1 \dot{\omega}_1, \lambda_2 \dot{\omega}_2, \lambda_3 \dot{\omega}_3) = -\underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\lambda_3 \dot{\omega}_3 = -(\omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1) = \omega_1 \omega_2 (\lambda_1 - \lambda_2)$$

$$\dot{\omega}_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3} \omega_1 \omega_2$$

$$\dot{\omega}_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1} \omega_2 \omega_3$$

$$\dot{\omega}_2 = \frac{\lambda_3 - \lambda_1}{\lambda_2} \omega_3 \omega_1$$

If $\lambda_1 = \lambda_2$ then $\dot{\omega}_3 = 0$

$$\dot{\omega}_1 = \boxed{\frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3} \omega_2 \equiv \Omega_b \omega_2$$

(Taylor 10.93 convention)

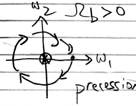
$$\dot{\omega}_2 = \boxed{\frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3} \omega_1 = -\Omega_b \omega_1$$

$$\ddot{\omega}_1 = -\Omega_b^2 \omega_1 \qquad \ddot{\omega}_2 = -\Omega_b^2 \omega_2$$

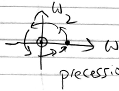
$$\omega_3 \equiv \omega_3 \qquad \omega_1 = \omega_0 \cos \Omega_b t \qquad \omega_2 = -\omega_0 \sin \Omega_b t$$

$$\dot{\omega}_1 = \Omega_b \omega_2 \qquad \dot{\omega}_2 = -\Omega_b \omega_1$$

$\Omega_b > 0$



precession $\sim -\hat{e}_3$



precession $\sim +\hat{e}_3$

The vector representing the precession of $\underset{\sim}{\omega}$ and of $\underset{\sim}{L}$ about $\hat{e}_3$ in the body frame is

$$\underset{\sim}{\Omega}_{body} = -\Omega_b \hat{e}_3 = -\frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \hat{e}_3$$

Writing (as we derived last time)

$$\underset{\sim}{\omega} = \frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3$$

(This shows that $\underset{\sim}{\omega}, \underset{\sim}{L}, \hat{e}_3$ are coplanar)

We get

$$\left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} = -\underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$= -\left(\frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3\right) \times \underset{\sim}{L}$$

$$\left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} = \left(-\Omega_b \hat{e}_3\right) \times \underset{\sim}{L}$$

$$= \underset{\sim}{\Omega}_{body} \times \underset{\sim}{L}$$

Since $\underset{\sim}{L}, \underset{\sim}{\omega}, \hat{e}_3$ are coplanar (for $\lambda_1 = \lambda_2$, $\underset{\sim}{\tau} = 0$)

$$\left(\frac{d\underset{\sim}{\omega}}{dt}\right)_{body} = \underset{\sim}{\Omega}_{body} \times \underset{\sim}{\omega}$$

Meanwhile
$$\left(\frac{d\hat{e}_3}{dt}\right)_{space} = 0 + \underset{\sim}{\omega} \times \hat{e}_3$$

$$= \left(\frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3\right) \times \hat{e}_3$$

$$= \left(\frac{\underset{\sim}{L}}{\lambda_1}\right) \times \hat{e}_3 = \underset{\sim}{\Omega}_{space} \times \hat{e}_3$$

also
$$\left(\frac{d\underset{\sim}{\omega}}{dt}\right)_{space} = \underset{\sim}{\Omega}_{space} \times \underset{\sim}{\omega}$$

We derived last time ($\lambda_1 = \lambda_2$, $\underset{\sim}{\tau} = 0$)

$$\frac{\underset{\sim}{L}}{\lambda_1} = \frac{\lambda_1 \omega_1 \hat{e}_1 + \lambda_1 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3}{\lambda_1} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{e}_3$$

$$= \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \underline{\omega_3 \hat{e}_3} + \frac{\lambda_3 \omega_3 \hat{e}_3}{\lambda_1} - \underline{\underline{\omega_3 \hat{e}_3}}$$

$$= \underset{\sim}{\omega} + \frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3 \hat{e}_3 = \underset{\sim}{\omega} + \underset{\sim}{\Omega}_{body}$$

$$= \underset{\sim}{\omega} - \Omega_b \hat{e}_3$$

$$\underset{\sim}{\Omega}_{space} = \frac{\underset{\sim}{L}}{\lambda_1} = \underset{\sim}{\omega} + \underset{\sim}{\Omega}_{body} = \underset{\sim}{\omega} - \Omega_b \hat{e}_3$$

$$\boldsymbol{\omega} \;=\; \omega_1\hat{\boldsymbol{e}}_1 + \omega_2\hat{\boldsymbol{e}}_2 + \omega_3\hat{\boldsymbol{e}}_3$$

symmetric top: $\lambda_1 = \lambda_2 \quad \Rightarrow \quad \boldsymbol{L} \;=\; \lambda_1\omega_1\hat{\boldsymbol{e}}_1 + \lambda_1\omega_2\hat{\boldsymbol{e}}_2 + \lambda_3\omega_3\hat{\boldsymbol{e}}_3$

$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \omega_1\hat{\boldsymbol{e}}_1 + \omega_2\hat{\boldsymbol{e}}_2 + \frac{\lambda_3}{\lambda_1}\omega_3\hat{\boldsymbol{e}}_3 \;=\; \omega_1\hat{\boldsymbol{e}}_1 + \omega_2\hat{\boldsymbol{e}}_2 + \omega_3\hat{\boldsymbol{e}}_3 + \frac{\lambda_3}{\lambda_1}\omega_3\hat{\boldsymbol{e}}_3 - \omega_3\hat{\boldsymbol{e}}_3$$

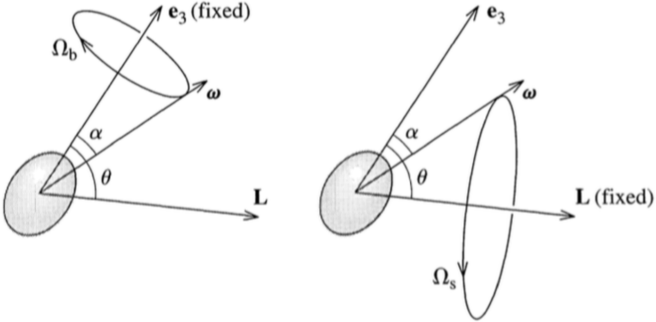
$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \boldsymbol{\omega} + \frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3\hat{\boldsymbol{e}}_3$$

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3\right)\hat{\boldsymbol{e}}_3 \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \Omega_b\,\hat{\boldsymbol{e}}_3$$

Last line proves that $\boldsymbol{\omega}$, $\boldsymbol{L}$, and $\hat{e}_3$ are coplanar (for $\lambda_1 = \lambda_2$).

Torque-free (10.94): $\boldsymbol{\omega} \;=\; \omega_0\cos(\Omega_b t)\hat{\boldsymbol{e}}_1 - \omega_0\sin(\Omega_b t)\hat{\boldsymbol{e}}_2 + \omega_3\hat{\boldsymbol{e}}_3$

Key trick for understanding “space” and “body” cones: decompose $\boldsymbol{\omega}$ into one part that points along $\boldsymbol{L}$ and one part that points along (or opposite) $\hat{e}_3$. [Sign of Ω_b depends on λ_1 vs. λ_3 magnitudes.]



(a) Body frame (b) Space frame

Figure 10.9 An axially symmetric body (shown here as a prolate spheroid or "egg-shaped" solid) is rotating with angular velocity $\boldsymbol{\omega}$, not in the direction of any of the principal axes. **(a)** As seen in the body frame, both $\boldsymbol{\omega}$ and $\mathbf{L}$ precess about the symmetry axis, $\mathbf{e}_3$, with angular frequency Ω_b given by (10.93). **(b)** As seen in the space frame, $\mathbf{L}$ is fixed, and both $\boldsymbol{\omega}$ and $\mathbf{e}_3$ precess about $\mathbf{L}$ with frequency Ω_s given by (10.96).

Torque-free precession of axially symmetric ($\lambda_1 = \lambda_2$) rigid body

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3 \quad \text{with} \quad \Omega_b \;=\; \frac{\lambda_1 - \lambda_3}{\lambda_1}\,\omega_3$$

$$\boldsymbol{\omega} \;=\; \omega_0 \cos(\Omega_b t)\hat{\boldsymbol{e}}_1 - \omega_0 \sin(\Omega_b t)\hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$$

$\boldsymbol{\Omega}_{\text{space}} = \boldsymbol{L}/\lambda_1$ points along $\boldsymbol{L}$. Describes precession of $\boldsymbol{\omega}$ (and $\hat{\boldsymbol{e}}_3$) about $\boldsymbol{L}$ as seen in space frame.

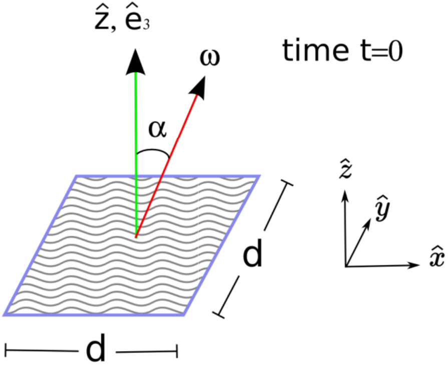
$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_3}{\mathrm{d}t} \;=\; \boldsymbol{\omega}\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1}\right)\times\hat{\boldsymbol{e}}_3 \;=\; \boldsymbol{\Omega}_{\text{space}}\times\hat{\boldsymbol{e}}_3$$

$\boxed{\boldsymbol{\Omega}_{\text{body}} = -\Omega_b \hat{\boldsymbol{e}}_3}$ points along $\hat{\boldsymbol{e}}_3$ if $\lambda_3 > \lambda_1$ (oblate, frisbee) and points opposite $\hat{\boldsymbol{e}}_3$ if $\lambda_3 < \lambda_1$ (prolate, US football). Describes precession of $\boldsymbol{\omega}$ (and $\boldsymbol{L}$) about $\hat{\boldsymbol{e}}_3$ as seen in body frame.

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{\text{body}} = -\boldsymbol{\omega}\times\boldsymbol{L} = -\left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\boldsymbol{L} = (-\Omega_b \hat{\boldsymbol{e}}_3)\times\boldsymbol{L} = \boldsymbol{\Omega}_{\text{body}}\times\boldsymbol{L}$$

$$\boldsymbol{\Omega}_{\text{space}} \;=\; \boldsymbol{\omega} \;+\; \boldsymbol{\Omega}_{\text{body}}$$

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



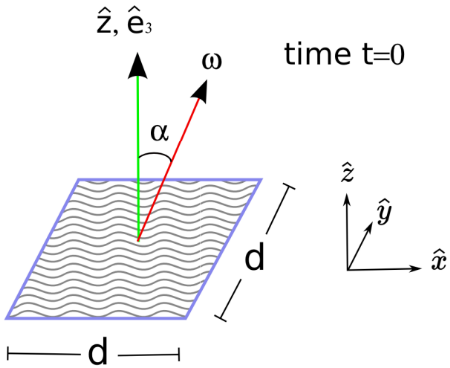
(a) Show that $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0.

(b) Calculate $\boldsymbol{L}$ at $t = 0$.

(c) Sketch $\hat{e}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$.

(d) Draw/label "body cone" and "space cone" on your sketch.

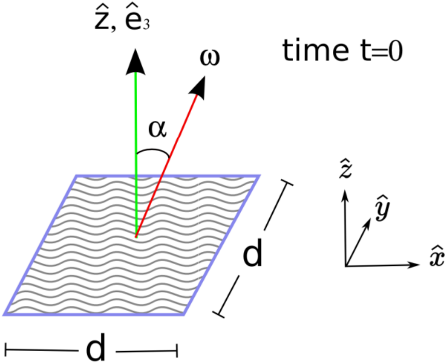
8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(e) Calculate precession frequencies Ω_{body} and Ω_{space}. Indicate directions of precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on drawing.

(f) You argue in HW that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(g) Find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α.

(h) When is this maximum deviation first reached?

video: https://www.youtube.com/watch?v=oH-dlrIF010

Just FYI, I put the final exams from 2015+2017 online at e.g.
`http://positron.hep.upenn.edu/p351/files/exam2015.pdf`

Let's work through Problem 1 together, which is the "prolate" (football-like) analogue of the "oblate" (frisbee-like) problem you'll work out in HW10.

Physics 351, Spring 2015, Final Exam.

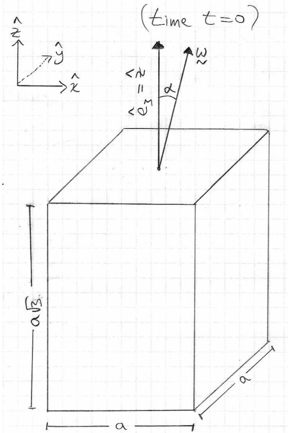
This closed-book exam has (only) 25% weight in your course grade. You can use one sheet of your own hand-written notes. Please show your work on these pages. The back side of each page is blank, so you can continue your work on the reverse side if you run out of space. Try to work in a way that makes your reasoning obvious to me, so that I can give you credit for correct reasoning even in cases where you might have made a careless error. Correct answers without clear reasoning may not receive full credit. Clear reasoning is especially important for "show that" questions.

The last page of the exam contains a list of equations that you might find helpful, to complement your own sheet of notes. You can detach it now if you like, before we begin.

The exam contains five questions, of equal weight. So each question is worth 20%. You might want to start with whichever questions you find easiest.

Because I believe that most of the learning in a physics course comes from your investing the time to work through homework problems, most of these exam problems are similar or identical to problems that you have already solved. The only point of the exams, in my opinion, is to motivate you to take the weekly homework seriously. So you should find nothing surprising in this exam.

Problem 1. A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\, a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\,\hat{\boldsymbol{z}} + \sin\alpha\,\hat{\boldsymbol{x}})$.



(a) Show (or argue) that the inertia tensor has the form $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and find the constant I_0.

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\boldsymbol{e}}_1$,$\hat{\boldsymbol{e}}_2$,$\hat{\boldsymbol{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

(c) Draw a sketch showing the vectors $\hat{\boldsymbol{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies $\Omega_{\rm body}$ and $\Omega_{\rm space}$. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\bf body}$ and $\boldsymbol{\Omega}_{\bf space}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\bf body}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\bf body}$ when you mean to draw $\boldsymbol{\Omega}_{\bf body}$.

(f) You argued in HW that $\mathbf{\Omega}_{\mathbf{space}} = \mathbf{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\mathbf{\Omega}_{\mathbf{space}}$ and $\mathbf{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{e}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

(h) At what time t is this maximum deviation first reached?

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

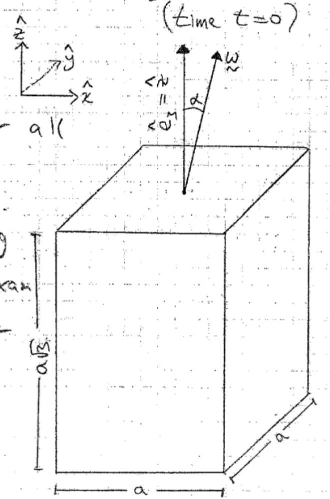
Problem 1.

A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\, a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{\mathbf{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\mathbf{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\mathbf{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.

(about the CoM)

(a) Show (or argue) that the inertia tensor has the form

$$\underline{\underline{I}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ and find the constant I_0.

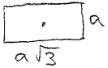


$I_{xy} = -\int xy\, dm = 0$ by symmetry, as for all off-diagonal elements.

$I_{zz} = \int (x^2+y^2)\,dm = \frac{m}{12}(a^2+a^2) = \frac{1}{6} m a^2$ using result for flat plate from back page of exam

$I_{xx} = \int (y^2+z^2)\,dm = \frac{m}{12}\left(a^2 + (\sqrt{3}a)^2\right) = \frac{1}{3} m a^2$

using flat plate



So $\boxed{I_0 = \frac{1}{6} m a^2}$

$\lambda_1 = \lambda_2 = 2I_0$, $\lambda_3 = I_0$

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\mathbf{e}}_1$,$\hat{\mathbf{e}}_2$,$\hat{\mathbf{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

$$\underset{\sim}{L} = \underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_3 \omega_3 \hat{e}_3 = 2 I_0 \omega \sin\alpha\, \hat{e}_1 + I_0 \omega \cos\alpha\, \hat{e}_3$$

since at $t=0$ $\hat{e}_1 = \hat{x}$, $\hat{e}_3 = \hat{z}$, $\underset{\sim}{\omega} = \omega \sin\alpha\, \hat{x} + \omega\cos\alpha\, \hat{z}$.

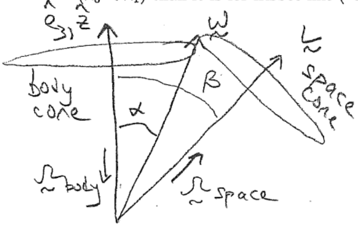
We can also write $\underset{\sim}{L} = 2 I_0 \omega \sin\alpha\, \hat{x} + I_0 \omega \cos\alpha\, \hat{z}$.

This second expression remains true for $t > 0$, because

$$0 = \underset{\sim}{\tau} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{\substack{\text{space}\\ \text{frame}}} \Rightarrow \underset{\sim}{L} \text{ is constant in space frame for torque-free rotation.}$$

The unit vectors $\hat{e}_i$ will rotate with the body. In particular, we will see L_1 and L_2 are not constant.

(c) Draw a sketch showing the vectors $\hat{\mathbf{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).



$$\tan\alpha = \frac{\omega_1}{\omega_3} \qquad \tan\beta = \frac{L_1}{L_3} = 2\tan\alpha$$

body cone is traced out by ω as it precesses about $\hat{e}_3$ in body frame.

space cone is traced out by ω as it precesses about L in space frame.

Note that L, ω, $\hat{e}_3$ remain coplanar!

$$\begin{aligned}
\hat{e}_3 &= 0\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \hat{e}_3 \\
L &= \lambda_1\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \lambda_3\omega_3\hat{e}_3 \\
\omega &= (\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \omega_3\hat{e}_3
\end{aligned}$$

coplanar: plane defined by $\hat{e}_3$ and $\omega_1\hat{e}_1 + \omega_2\hat{e}_2 = \vec{\omega}_\perp$

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\mathbf{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\mathbf{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\mathbf{body}}$.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\text{body}}$ and $\boldsymbol{\Omega}_{\text{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\text{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\text{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\text{body}}$.

$$\underset{\sim}{\Omega}_{\text{space}} = \frac{\underset{\sim}{L}}{\lambda_1} = \omega \sin\alpha\, \hat{x} + \frac{\lambda_3}{\lambda_1}\omega\cos\alpha\, \hat{z} = \omega\left(\sin\alpha\, \hat{x} + \tfrac{1}{2}\cos\alpha\, \hat{z}\right)$$

$$= \left(\frac{\omega}{2}\sqrt{\cos^2\alpha + 4\sin^2\alpha}\right)\hat{L} = \left(\frac{\omega}{2}\sqrt{1+3\sin^2\alpha}\right)\hat{L}$$

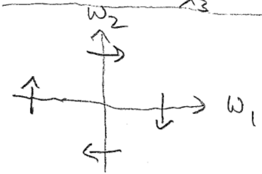
At $t=0$, $\underset{\sim}{\Omega}_{\text{space}} = \omega\sin\alpha\, \hat{e}_1 + \frac{1}{2}\omega\cos\alpha\, \hat{e}_3$

$$\underset{\sim}{\Omega}_{\text{body}} = \frac{\lambda_3-\lambda_1}{\lambda_1}\omega_3\hat{e}_3 = \frac{I_0 - 2I_0}{2I_0}\omega_3\hat{e}_3 = -\frac{1}{2}\omega\cos\alpha\, \hat{e}_3$$

$$\boxed{\dot{\omega}_3 = \omega_1\omega_2\frac{\lambda_1-\lambda_2}{\lambda_3} = 0}$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1} = \frac{1}{2}\omega_3\,\omega_2 = \left(\frac{1}{2}\omega\cos\alpha\right)\omega_2$$

$$\dot{\omega}_2 = \omega_3\omega_1\frac{\lambda_3-\lambda_1}{\lambda_2} = -\frac{1}{2}\omega_3\,\omega_1 = -\left(\frac{1}{2}\omega\cos\alpha\right)\omega_1$$



clockwise $\rightarrow$ $\underset{\sim}{\omega}$ precesses in $-\hat{e}_3$ direction in body frame

(f) You argued in HW11 that $\mathbf{\Omega}_{\mathbf{space}} = \mathbf{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\mathbf{\Omega}_{\mathbf{space}}$ and $\mathbf{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

At $t=0$, $\underset{\sim}{\Omega}_{body} = -\frac{1}{2}\omega\cos\alpha\,\hat{e}_3$

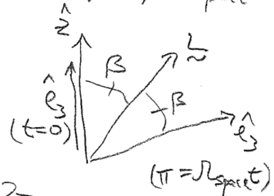
$\underset{\sim}{\omega} = \omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

add to $\quad +\frac{1}{2}\omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

At $t=0$, $\underset{\sim}{\Omega}_{space} = \omega\sin\alpha\,\hat{e}_1 + \frac{1}{2}\omega\cos\alpha\,\hat{e}_3$ ✓

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

The initial angle between $\hat{e}_3$ and $\underset{\sim}{L}$ is $\beta = \operatorname{atan}(2\tan\alpha) \simeq 2\alpha$.
This angle is a constant of the motion, because $L_3 = \lambda_3 \omega_3 = \text{const.}$ and $\underset{\sim}{L}_{\text{space}}$ (hence magnitude of $\underset{\sim}{L}_{\text{body}}$) is constant. As shown on diagram (c), $\underset{\sim}{L}, \underset{\sim}{\omega}, \hat{e}_3$ remain coplanar. So in the space frame, $\underset{\sim}{\omega}$ and $\hat{e}_3$ both precess about $\underset{\sim}{L}$ with frequency Ω_{space}.
Maximum angle between $\hat{e}_3$ and $\hat{z}$ is $\boxed{2\beta \simeq 4\alpha}$.



(h) At what time t is this maximum deviation first reached?

one-half period of Ω_{space}:

$$\Omega_{\text{space}} t = \pi$$

$$t = \frac{\pi}{\Omega_{\text{space}}} = \frac{2\pi}{\omega\sqrt{1+3\sin^2\alpha}} \simeq \frac{2\pi}{\omega} \quad \left(\simeq \frac{2\pi}{\omega_3} \text{ if } \alpha \ll 1\right)$$

So the precession ("wobble") has $\simeq$ half the frequency of the "spin" if $\alpha \ll 1$.

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

(Taylor 10.35) A rigid body consists of:

$$\begin{aligned} m \text{ at } (a,0,0) &= a(1,\ 0,\ \ 0) \\ 2m \text{ at } (0,a,a) &= a(0,\ 1,\ \ 1) \\ 3m \text{ at } (0,a,-a) &= a(0,\ 1,-1) \end{aligned}$$

Find inertia tensor $\underline{\underline{\boldsymbol{I}}}$, its principal moments, and the principal axes.

$$I_{xx} = \sum m(y^2+z^2) = ma^2(2\cdot2+2\cdot3) = 10ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{xy} = -\sum mxy = -ma^2(0) = 0$$

$$I_{xz} = -\sum mxz = -ma^2(0) = 0$$

$$I_{yz} = -\sum myz = -ma^2(2-3) = ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} ma^2$$

$$\underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda\,\underset{\sim}{\omega} \Rightarrow \left(\underline{\underline{I}} - \lambda\,\mathbb{1}\right)\underset{\sim}{\omega} = 0$$

$$\Rightarrow \det\left(\underline{\underline{I}} - \lambda\,\mathbb{1}\right) = 0$$

$$0 = (10-\lambda)(6-\lambda)^2 - (10-\lambda) \Rightarrow \lambda = 10 \text{ or } (6-\lambda)^2 = 1$$

$$6-\lambda = 1 \Rightarrow \lambda = 5, \quad 6-\lambda = -1 \Rightarrow \lambda = 7 \qquad \lambda \in \{10, 7, 5\}$$

$$\lambda \in \{10, 7, 5\}\, ma^2$$

$$0 = \begin{pmatrix} 10-10 & 0 & 0 \\ 0 & 6-10 & 1 \\ 0 & 1 & 6-10 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4y+z \\ y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = z_1 = 0 \Rightarrow \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-7 & 0 & 0 \\ 0 & 6-7 & 1 \\ 0 & 1 & 6-7 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3x \\ z-y \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0,\ y_2 = z_2 \Rightarrow \hat{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-5 & 0 & 0 \\ 0 & 6-5 & 1 \\ 0 & 1 & 6-5 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5x \\ y+z \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3 = 0,\ y_3 = -z_3 \Rightarrow \hat{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

WolframAlpha

eigenvectors {{10,0,0},{0,6,1},{0,1,6}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}\right]$$

Results:

$v_1 = (1, 0, 0)$

$v_2 = (0, 1, 1)$

$v_3 = (0, -1, 1)$

Corresponding eigenvalues:

$\lambda_1 = 10$

$\lambda_2 = 7$

$\lambda_3 = 5$

```
In[1]:= m = {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

Out[1]= {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

In[2]:= MatrixForm[m]

Out[2]//MatrixForm=
```

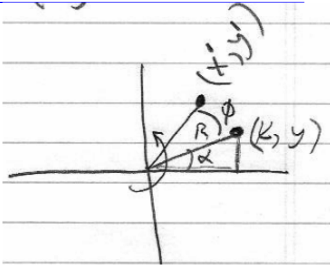
$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

```
In[3]:= Eigenvalues[m]

Out[3]= {10, 7, 5}

In[4]:= Eigenvectors[m]

Out[4]= {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}

In[5]:= Eigensystem[m]

Out[5]= {{10, 7, 5},
         {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}}
```

One useful tool for relating the fixed $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ axes to the rigid body's $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$, axes is the "Euler angles," ϕ, θ, ψ.

(Another way, which I used in the simulation program for the struck triangle, is simply to keep track instant-by-instant of the x, y, z components of $\hat{\boldsymbol{e}}_1(t)$, $\hat{\boldsymbol{e}}_2(t)$, $\hat{\boldsymbol{e}}_3(t)$. But if you're given the three Euler angles, you can compute the x, y, z components of the body axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$.)

Question: Suppose I rotate the vector $(x, y) = R(\cos\alpha, \sin\alpha)$ by an angle ϕ (about the origin). How would you write x' as a linear combination of x and y? How about y' as a linear combination of x and y?

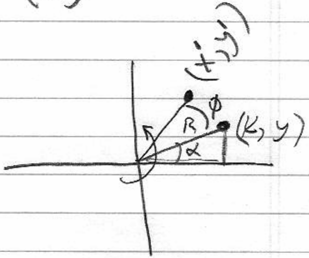


Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = x\cos\phi - y\sin\phi$$

$$y' = x\sin\phi + y\cos\phi$$



$$(x, y) = R(\cos\alpha, \sin\alpha)$$

$$(x', y') = R\big(\cos(\alpha+\phi), \sin(\alpha+\phi)\big)$$

$$= R\big(\cos\alpha\cos\phi - \sin\alpha\sin\phi, \ \sin\alpha\cos\phi + \cos\alpha\sin\phi\big)$$

$$= \big(x\cos\phi - y\sin\phi, \ y\cos\phi + x\sin\phi\big)$$

Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

x', y' as above; $z' = z$

Rotate by angle θ about $\hat{y}'$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

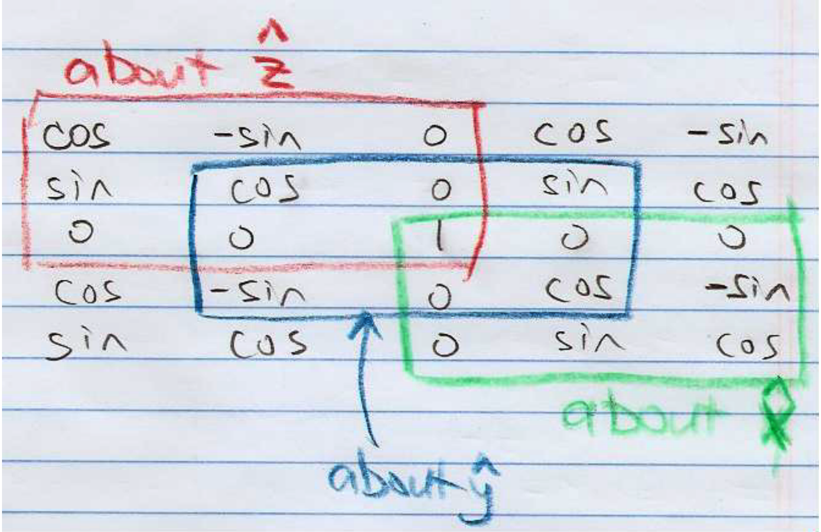
Mnemonic: for infinitessimal rotation angle $\epsilon \ll 1$, $\boldsymbol{r} \to \boldsymbol{r} + \epsilon\hat{\boldsymbol{\omega}} \times \boldsymbol{r}$. So for rotation about $\hat{\boldsymbol{y}}$, $(1,0,0) \to (1,0,-\epsilon)$, since $\epsilon\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\epsilon\hat{\boldsymbol{z}}$.

The hardest part of writing down 3×3 rotation matrices is remembering where to put the minus sign.

$$\begin{matrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{matrix}$$

Once you've worked out one case correctly (e.g. from a diagram), here's a trick (thanks to 2015 student Adam Zachar) for working out the other two . . .

Just add two more columns and two more rows, following the cycles: xyz, yzx, zxy. Then draw boxes of size 3×3.



(Check previous result using Mathematica.)

```
In[1]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & -\mathrm{Sin}[\phi] & 0 \\ \mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[2]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] \\ 0 & 1 & 0 \\ -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix}$$

```
In[3]:= RotationMatrix[α, {1, 0, 0}] // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] \\ 0 & \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] \end{pmatrix}$$

Euler angles: can move (x, y, z) axes to arbitrary orientation.

Rotate by ϕ about $\hat{z}$

Then rotate by θ about $\hat{y}'$ $(\hat{e}_2')$

Then rotate by ψ about $\hat{z}''$ $(\hat{e}_3)$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \left(\begin{array}{c|c|c} c\theta c\phi c\psi - s\phi s\psi & -c\theta s\phi c\psi - c\phi s\psi & s\theta c\psi \\ c\theta c\phi s\psi + s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\psi \\ -s\theta c\phi & s\theta s\phi & c\theta \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

```
In[2]:= RotationMatrix[ϕ] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix}$$

```
In[4]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] & 0 \\ \text{Sin}[\phi] & \text{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[5]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix}$$

```
In[10]:= r1 = RotationMatrix[ϕ, {0, 0, 1}];
         r2 = RotationMatrix[θ, {0, 1, 0}];
         r3 = RotationMatrix[ψ, {0, 0, 1}];
         r3 . r2 . r1 // MatrixForm
```

Out[13]//MatrixForm=

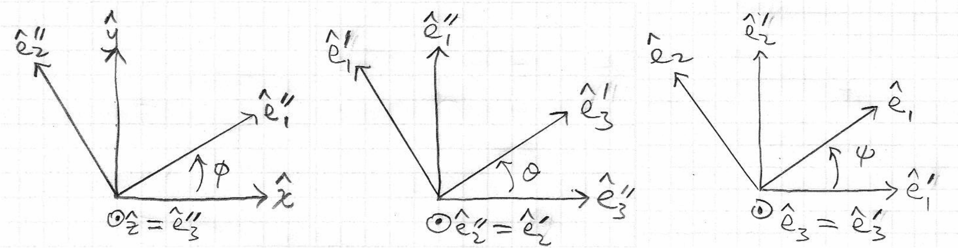
$$\begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Sin}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta] \\ \text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\psi] \\ -\text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\theta] \end{pmatrix}$$

Let the Euler angles ϕ, θ, ψ vary with time, as body rotates.

I'll write out more steps than Taylor does, and I may confuse you by saying $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \to (\hat{\boldsymbol{e}}_1'', \hat{\boldsymbol{e}}_2'', \hat{\boldsymbol{e}}_3'') \to (\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3') \to (\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$.

I do this so that my $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$ are the same as Taylor's.

1. Rotate by ϕ about $\hat{\boldsymbol{z}} \to \hat{\boldsymbol{e}}_1''$, $\hat{\boldsymbol{e}}_2''$. ($\hat{\boldsymbol{e}}_3'' = \hat{\boldsymbol{z}}$.)
2. Rotate by θ about $\hat{\boldsymbol{e}}_2'' \to \hat{\boldsymbol{e}}_1'$, $\hat{\boldsymbol{e}}_3'$. ($\hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_2''$.)
3. Rotate by ψ about $\hat{\boldsymbol{e}}_3' \to \hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$. ($\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_3'$.)



$$\boldsymbol{\omega} \;=\; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2'' + \dot{\psi}\,\hat{\boldsymbol{e}}_3' \;=\; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$$

Remarkable trick: We can write $\boldsymbol{\omega}$ as vector sum of 3 separate angular-velocity vectors, about three successive axes.

Next, project $\boldsymbol{\omega}$ onto more convenient sets of unit vectors.

(Skip this — here for reference)

(orthogonal matrix: inverse = transpose)

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\underset{\sim}{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' = \dot{\phi}\hat{z} + \dot{\theta}(-\hat{x}\sin\phi + \hat{y}\cos\phi) + \dot{\psi}(\hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi, +\dot{\theta}\cos\phi, \dot{\phi}) + \dot{\psi}\left(\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \cos\theta(\hat{z})\right)$$

$$\underset{\sim}{\omega} = \left(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi,\ \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,\ \dot{\phi} + \dot{\psi}\cos\theta\right)$$

(IN SPACE AXES)

$$\left.\begin{aligned} \hat{e}_1'' &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{e}_2'' &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{e}_3'' &= \hat{z} \end{aligned}\right\} \quad \begin{aligned} \hat{x} &= \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi \\ \hat{y} &= \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1' &= \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \\ \hat{e}_3' &= \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \\ \hat{e}_2' &= \hat{e}_2'' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1'' &= \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta \\ \hat{e}_3'' &= -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1 &= \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \\ \hat{e}_2 &= -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \\ \hat{e}_3 &= \hat{e}_3' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1' &= \hat{e}_1\cos\psi - \hat{e}_2\sin\psi \\ \hat{e}_2' &= \hat{e}_1\sin\psi + \hat{e}_2\cos\psi \end{aligned}$$

Start from $\boldsymbol{\omega} = \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$ and substitute preferred unit vectors.

In the "space" basis [proof on previous page]:

$$\boldsymbol{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\hat{\boldsymbol{x}} + (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{\boldsymbol{z}}$$

In the "body" basis [proof on next page]:

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{\boldsymbol{e}}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{\boldsymbol{e}}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$): in the "primed" basis (i.e. before the final rotation by ψ about $\hat{\boldsymbol{e}}_3$). Note that $\hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_3$.

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' + (\dot{\theta})\hat{\boldsymbol{e}}_2' + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

This last one is easiest to see if you consider the instant at which $\psi = 0$.

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\begin{aligned}
\underset{\sim}{\omega} &= \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\hat{e}_3'' + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\left(-\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta\right) + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1'\right) + \dot{\phi}\cos\theta\left(\hat{e}_3'\right)\right] + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1\cos\psi - \hat{e}_2\sin\psi\right) + \dot{\phi}\cos\theta\,\hat{e}_3\right] + \dot{\theta}\left(\hat{e}_1\sin\psi + \hat{e}_2\cos\psi\right) + \dot{\psi}\hat{e}_3' \\
&= \hat{e}_1\left(-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right) + \hat{e}_2\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + \hat{e}_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)
\end{aligned}$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$):

$$\boldsymbol{\omega} \;=\; (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

This basis makes it easy to write down the top's angular momentum $\boldsymbol{L}$, kinetic energy T, and Lagrangian $\mathcal{L}$.

$$\boldsymbol{L} \;=\; (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

We then find two ignorable coordinates: ϕ and ψ. So using the corresponding conserved quantities, we can reduce the θ EOM to a single-variable problem.

$$\boldsymbol{\omega} \;=\; (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$\boldsymbol{L} \;=\; (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)} \equiv P_\psi \equiv \text{const.} = \lambda_3\omega_3 = \boxed{L_3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta \equiv P_\phi \equiv \text{const.}$$

$$= \boxed{(L_z - L_3\cos\theta)} + \boxed{L_3}\cos\theta = L_z$$

Digression: $\hat{z} = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$

$$L_z = \underset{\sim}{L}\cdot\hat{z} = \lambda_1\dot{\phi}\sin^2\theta + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta$$

$$L_z = \lambda_1\dot{\phi}\sin^2\theta + L_3\cos\theta$$

$$\Rightarrow \lambda_1\dot{\phi}\sin^2\theta = L_z - L_3\cos\theta$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

θ equation of motion: (derivation 2 pages ahead)

$$\lambda_1 \ddot\theta = \lambda_1 \boxed{\dot\phi^2} \sin\theta\cos\theta - (\lambda_3\omega_3)\boxed{\dot\phi}\sin\theta + MgR\sin\theta$$

First consider case where θ = constant. Since
$\lambda_1 \dot\phi \sin^2\theta = L_z - L_3\cos\theta$, $\theta = \text{const} \Rightarrow \dot\phi = \text{const} \equiv \Omega$

$$0 = \lambda_1 \boxed{\Omega^2}\cancel{\sin\theta}\cos\theta - \lambda_3\omega_3\boxed{\Omega}\cancel{\sin\theta} + MgR\cancel{\sin\theta}$$

$$(\lambda_1\cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR = 0$$

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1\cos\theta MgR}}{2\lambda_1\cos\theta}$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1\cos\theta MgR}{(\lambda_3\omega_3)^2}}\right)$$

$$\Omega = \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1 \cos\theta M g R}{(\lambda_3\omega_3)^2}}\right)$$

has 2 real solutions if $(\lambda_3\omega_3)^2 > 4\lambda_1 M g R\cos\theta$
("if ω_3 is large enough")

Math is simplest if $(\lambda_3\omega_3)^2 \gg 4\lambda_1 M g R\cos\theta$
("ω_3 is very large")

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \left(1 - \frac{2\lambda_1 M g R\cos\theta}{(\lambda_3\omega_3)^2}\right)\right)$$

$$\Omega_+ \simeq \frac{\lambda_3\omega_3}{\lambda_1\cos\theta}$$

$$\Omega_- \simeq \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\cdot\frac{2\lambda_1 M g R\cos\theta}{(\lambda_3\omega_3)^2} = \boxed{\frac{MgR}{\lambda_3\omega_3} = \Omega_-}$$

This is Ω_{space} for tossed football with no torque

(See HW)

This can be seen from

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$MgR\sin\theta = \lambda_3\omega_3\,\Omega\sin\theta$$

(Skip: Just in case you wanted to see the θ EOM derived.)

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})(-\dot{\phi}\sin\theta) + MgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\lambda_1 \dot{\theta}) = \lambda_1 \ddot{\theta}$$

$$\lambda_1 \ddot{\theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta - \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}\sin\theta + MgR\sin\theta$$

$$\lambda_1 \ddot{\theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta - \lambda_3\omega_3\dot{\phi}\sin\theta + MgR\sin\theta$$

$$\lambda_1 \dot\phi \sin^2\theta = L_z - L_3\cos\theta \Rightarrow \dot\phi = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$$

$$E = T + U = \frac{1}{2}\lambda_1\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}\lambda_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 + MgR\cos\theta$$

$$E = \frac{1}{2}\lambda_1\sin^2\theta\,\dot\phi^2 + \frac{1}{2}\lambda_1\dot\theta^2 + \frac{1}{2}\lambda_3\omega_3^2 + MgR\cos\theta$$

$$E = \frac{\lambda_1\sin^2\theta}{2}\cdot\frac{(L_z - L_3\cos\theta)^2}{\lambda_1^2\sin^4\theta} + \frac{\lambda_1\dot\theta^2}{2} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

$$E = \frac{\lambda_1\dot\theta^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2}\lambda_1\dot\theta^2 + U_{eff}(\theta)$$

(one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2

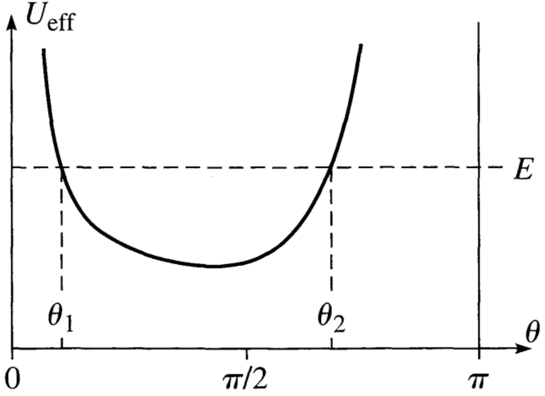
$$E = \frac{\lambda_1 \dot{\theta}^2}{2} + \boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}$$

$U_{eff}(\theta)$

$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$

(one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2



Physics 351 — Wednesday, April 4, 2018

- Pick up **worksheet (!)** on your way in.
- HW10 due this Friday. I tried (!) to make it short. For HW help, Bill is in DRL 3N6 Wed 4–7pm. Grace is in DRL 2C2 Thu 5:30–8:30pm. To get the most benefit from the homework, first work through every problem on your own to the best of your ability. Then check in with me, Grace, or a friend to compare final results and to trade suggestions on problems that stumped you.
- What if in future years I did weekly quizzes (simple repeat of a HW problem, after HW has been graded & handed back with solutions), but scaled them `scaled = min(raw/0.66,1.00)` so that earning 2/3 of semester quiz points earns you 100%?
- Let's start today by trying to restate more clearly Monday's discussion of $\mathbf{\Omega}_{\text{space}}$ and $\mathbf{\Omega}_{\text{body}}$.