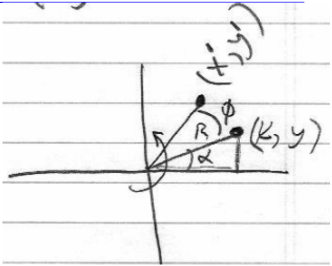
Physics 351 — Friday, April 6, 2018

- Turn in HW10 either today or Monday, as you prefer. Grace will offer an extra help session this Sunday (4/8), 1pm–3pm in DRL 3W2.
- Pick up handout for HW11, due next Friday.
- This weekend, read Taylor's chapter 13 (Hamiltonian mechanics), which we'll start to discuss on Wednesday. Hamiltonians will be the last major topic of the semester.
- We'll spend Monday on coupled oscillators (chapter 11), so if you didn't read it last weekend, please do so now.

One useful tool for relating the fixed $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ axes to the rigid body's $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$, axes is the "Euler angles," ϕ, θ, ψ.

(Another way, which I used in the simulation program for the struck triangle (from a few days ago), is simply to keep track instant-by-instant of the x, y, z components of $\hat{\boldsymbol{e}}_1(t)$, $\hat{\boldsymbol{e}}_2(t)$, $\hat{\boldsymbol{e}}_3(t)$. But if you're given the three Euler angles, you can compute the x, y, z components of the body axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$.)

Question: Suppose I rotate the vector $(x, y) = R(\cos\alpha, \sin\alpha)$ by an angle ϕ (about the origin). How would you write x' as a linear combination of x and y? How about y' as a linear combination of x and y?

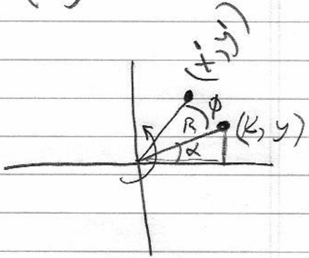


Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = x\cos\phi - y\sin\phi$$

$$y' = x\sin\phi + y\cos\phi$$



$$(x, y) = R(\cos\alpha, \sin\alpha)$$

$$(x', y') = R\big(\cos(\alpha+\phi), \sin(\alpha+\phi)\big)$$

$$= R\big(\cos\alpha\cos\phi - \sin\alpha\sin\phi, \ \sin\alpha\cos\phi + \cos\alpha\sin\phi\big)$$

$$= \big(x\cos\phi - y\sin\phi, \ y\cos\phi + x\sin\phi\big)$$

Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

x', y' as above; $z' = z$

Rotate by angle θ about $\hat{y}'$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

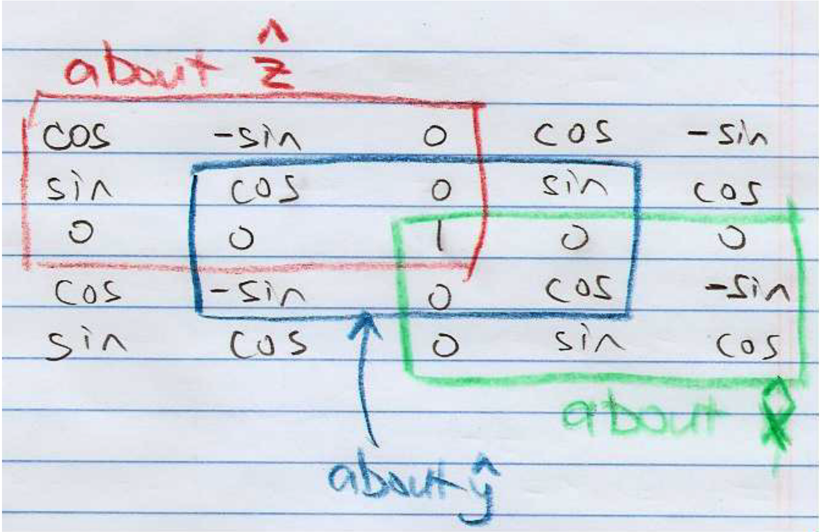
Mnemonic: for infinitessimal rotation angle $\epsilon \ll 1$, $\boldsymbol{r} \to \boldsymbol{r} + \epsilon\hat{\boldsymbol{\omega}} \times \boldsymbol{r}$. So for rotation about $\hat{\boldsymbol{y}}$, $(1,0,0) \to (1,0,-\epsilon)$, since $\epsilon\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\epsilon\hat{\boldsymbol{z}}$.

The hardest part of writing down 3×3 rotation matrices is remembering where to put the minus sign.

$$\begin{matrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{matrix}$$

Once you've worked out one case correctly (e.g. from a diagram), here's a trick (thanks to 2015 student Adam Zachar) for working out the other two . . .

Just add two more columns and two more rows, following the cycles: xyz, yzx, zxy. Then draw boxes of size 3×3.



(Check previous result using Mathematica.)

```
In[1]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & -\mathrm{Sin}[\phi] & 0 \\ \mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[2]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] \\ 0 & 1 & 0 \\ -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix}$$

```
In[3]:= RotationMatrix[α, {1, 0, 0}] // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] \\ 0 & \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] \end{pmatrix}$$

Euler angles: can move (x, y, z) axes to arbitrary orientation.

Rotate by ϕ about $\hat{z}$

Then rotate by θ about $\hat{y}'$ $(\hat{e}_2')$

Then rotate by ψ about $\hat{z}''$ $(\hat{e}_3)$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \left(\begin{array}{c|c|c} c\theta c\phi c\psi - s\phi s\psi & -c\theta s\phi c\psi - c\phi s\psi & s\theta c\psi \\ c\theta c\phi s\psi + s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\psi \\ -s\theta c\phi & s\theta s\phi & c\theta \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

```
In[2]:= RotationMatrix[ϕ] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix}$$

```
In[4]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] & 0 \\ \text{Sin}[\phi] & \text{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[5]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix}$$

```
In[10]:= r1 = RotationMatrix[ϕ, {0, 0, 1}];
         r2 = RotationMatrix[θ, {0, 1, 0}];
         r3 = RotationMatrix[ψ, {0, 0, 1}];
         r3 . r2 . r1 // MatrixForm
```

Out[13]//MatrixForm=

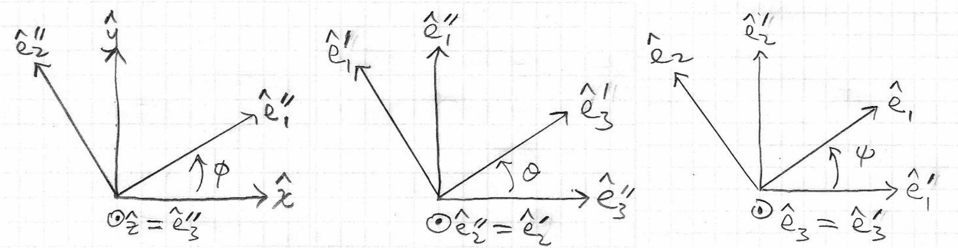
$$\begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Sin}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta] \\ \text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\psi] \\ -\text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\theta] \end{pmatrix}$$

Let the Euler angles ϕ, θ, ψ vary with time, as body rotates.

I'll write out more steps than Taylor does, and I may confuse you by saying $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \to (\hat{\boldsymbol{e}}_1'', \hat{\boldsymbol{e}}_2'', \hat{\boldsymbol{e}}_3'') \to (\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3') \to (\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$.

I do this so that my $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$ are the same as Taylor's.

1. Rotate by ϕ about $\hat{\boldsymbol{z}} \to \hat{\boldsymbol{e}}_1''$, $\hat{\boldsymbol{e}}_2''$. $(\hat{\boldsymbol{e}}_3'' = \hat{\boldsymbol{z}}.)$
2. Rotate by θ about $\hat{\boldsymbol{e}}_2'' \to \hat{\boldsymbol{e}}_1'$, $\hat{\boldsymbol{e}}_3'$. $(\hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_2''.)$
3. Rotate by ψ about $\hat{\boldsymbol{e}}_3' \to \hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$. $(\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_3'.)$



$$\boldsymbol{\omega} \; = \; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2'' + \dot{\psi}\,\hat{\boldsymbol{e}}_3' \; = \; \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$$

Remarkable trick: We can write $\boldsymbol{\omega}$ as vector sum of 3 separate angular-velocity vectors, about three successive axes.

Next, project $\boldsymbol{\omega}$ onto more convenient sets of unit vectors.

(Skip this — here for reference)

(orthogonal matrix: inverse = transpose)

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\underset{\sim}{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' = \dot{\phi}\hat{z} + \dot{\theta}(-\hat{x}\sin\phi + \hat{y}\cos\phi) + \dot{\psi}(\hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi, +\dot{\theta}\cos\phi, \dot{\phi}) + \dot{\psi}\left(\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \cos\theta(\hat{z})\right)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi,\ \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,\ \dot{\phi} + \dot{\psi}\cos\theta)$$

(IN SPACE AXES)

$$\left.\begin{aligned} \hat{e}_1'' &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{e}_2'' &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{e}_3'' &= \hat{z} \end{aligned}\right\} \quad \begin{aligned} \hat{x} &= \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi \\ \hat{y} &= \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1' &= \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \\ \hat{e}_3' &= \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \\ \hat{e}_2' &= \hat{e}_2'' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1'' &= \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta \\ \hat{e}_3'' &= -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1 &= \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \\ \hat{e}_2 &= -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \\ \hat{e}_3 &= \hat{e}_3' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1' &= \hat{e}_1\cos\psi - \hat{e}_2\sin\psi \\ \hat{e}_2' &= \hat{e}_1\sin\psi + \hat{e}_2\cos\psi \end{aligned}$$

Start from $\boldsymbol{\omega} = \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$ and substitute preferred unit vectors.

In the "space" basis [proof on previous page]:

$$\boldsymbol{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\hat{\boldsymbol{x}} + (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{\boldsymbol{z}}$$

In the "body" basis [proof on next page]:

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{\boldsymbol{e}}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{\boldsymbol{e}}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$): in the "primed" basis (i.e. before the final rotation by ψ about $\hat{\boldsymbol{e}}_3$). Note that $\hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_3$.

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' + (\dot{\theta})\hat{\boldsymbol{e}}_2' + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

This last one is easiest to see if you consider the instant at which $\psi = 0$.

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$

$$\begin{aligned}
\underset{\sim}{\omega} &= \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\hat{e}_3'' + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' \\
&= \dot{\phi}\left(-\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta\right) + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1'\right) + \dot{\phi}\cos\theta\left(\hat{e}_3'\right)\right] + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right) \\
&= \left[-\dot{\phi}\sin\theta\left(\hat{e}_1\cos\psi - \hat{e}_2\sin\psi\right) + \dot{\phi}\cos\theta\,\hat{e}_3\right] + \dot{\theta}\left(\hat{e}_1\sin\psi + \hat{e}_2\cos\psi\right) + \dot{\psi}\hat{e}_3' \\
&= \hat{e}_1\left(-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right) + \hat{e}_2\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + \hat{e}_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)
\end{aligned}$$

Most convenient for **symmetric** top ($\lambda_1 = \lambda_2$):

$$\boldsymbol{\omega} \;=\; (-\dot\phi\sin\theta)\hat{\boldsymbol{e}}'_1 \;+\; (\dot\theta)\hat{\boldsymbol{e}}'_2 \;+\; (\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}'_3$$

This basis makes it easy to write down the top's angular momentum $\boldsymbol{L}$, kinetic energy T, and Lagrangian $\mathcal{L}$. (Try it!)

$$\boldsymbol{L} \;=\; (-\lambda_1\dot\phi\sin\theta)\hat{\boldsymbol{e}}'_1 \;+\; (\lambda_1\dot\theta)\hat{\boldsymbol{e}}'_2 \;+\; \lambda_3(\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}'_3$$

$$T = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

We then find two ignorable coordinates: ϕ and ψ. So using the corresponding conserved quantities, we can reduce the θ EOM to a single-variable problem.

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}'_1 + (\dot{\theta})\hat{\boldsymbol{e}}'_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}'_3$$

$$\boldsymbol{L} = (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}'_1 + (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}'_2 + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}'_3$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)} \equiv P_\psi \equiv \text{const.} = \lambda_3\omega_3 = \boxed{L_3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta \equiv P_\phi \equiv \text{const.}$$

$$= \boxed{(L_z - L_3\cos\theta)} + \boxed{L_3}\cos\theta = L_z$$

Digression: $\hat{z} = -\hat{e}'_1\sin\theta + \hat{e}'_3\cos\theta$

$$L_z = \underset{\sim}{L}\cdot\hat{z} = \lambda_1\dot{\phi}\sin^2\theta + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta$$

$$L_z = \lambda_1\dot{\phi}\sin^2\theta + L_3\cos\theta$$

$$\Rightarrow \lambda_1\dot{\phi}\sin^2\theta = L_z - L_3\cos\theta$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

θ equation of motion: (derivation 2 pages ahead)

$$\lambda_1 \ddot{\theta} = \lambda_1 \boxed{\dot{\phi}^2} \sin\theta\cos\theta - (\lambda_3\omega_3)\boxed{\dot{\phi}} \sin\theta + MgR\sin\theta$$

First consider case where $\theta =$ constant. Since $\lambda_1 \dot{\phi}\sin^2\theta = L_z - L_3\cos\theta$, $\theta = \text{const} \Rightarrow \dot{\phi} = \text{const} \equiv \Omega$

$$0 = \lambda_1 \boxed{\Omega^2} \cancel{\sin\theta}\cos\theta - \lambda_3\omega_3 \boxed{\Omega} \cancel{\sin\theta} + MgR\cancel{\sin\theta}$$

$$(\lambda_1 \cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR = 0$$

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1\cos\theta MgR}}{2\lambda_1\cos\theta}$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1\cos\theta MgR}{(\lambda_3\omega_3)^2}}\right)$$

$$\Omega = \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1 \cos\theta M g R}{(\lambda_3 \omega_3)^2}}\right)$$

has 2 real solutions if $(\lambda_3 \omega_3)^2 > 4\lambda_1 M g R \cos\theta$
("if ω_3 is large enough")

Math is simplest if $(\lambda_3\omega_3)^2 \gg 4\lambda_1 M g R \cos\theta$
("ω_3 is very large")

$$\Omega = \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\theta}\left(1 \pm \left(1 - \frac{2\lambda_1 M g R \cos\theta}{(\lambda_3 \omega_3)^2}\right)\right)$$

$$\Omega_+ \simeq \frac{\lambda_3\omega_3}{\lambda_1 \cos\theta} \qquad \Omega_- \simeq \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta} \cdot \frac{2\lambda_1 M g R \cos\theta}{(\lambda_3\omega_3)^2} = \boxed{\frac{MgR}{\lambda_3\omega_3} = \Omega_-}$$

This is Ω_{space} for tossed football with no torque

(See HW)

This can be seen from

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$MgR\sin\theta = \lambda_3 \omega_3 \, \Omega \sin\theta$$

(Skip: Just in case you wanted to see the θ EOM derived.)

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial\theta} = \lambda_1\dot\phi^2 \sin\theta\cos\theta + \lambda_3(\dot\phi\cos\theta + \dot\psi)(-\dot\phi\sin\theta) + MgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\dot\theta}\right) = \frac{d}{dt}(\lambda_1\dot\theta) = \lambda_1\ddot\theta$$

$$\lambda_1\ddot\theta = \lambda_1\dot\phi^2\sin\theta\cos\theta - \lambda_3(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta + MgR\sin\theta$$

$$\lambda_1\ddot\theta = \lambda_1\dot\phi^2\sin\theta\cos\theta - \lambda_3\omega_3\dot\phi\sin\theta + MgR\sin\theta$$

$$\lambda_1 \dot{\phi} \sin^2\theta = L_z - L_3\cos\theta \Rightarrow \dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$$

$$E = T + U = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + MgR\cos\theta$$

$$E = \frac{1}{2}\lambda_1 \sin^2\theta\,\dot{\phi}^2 + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3 \omega_3^2 + MgR\cos\theta$$

$$E = \frac{\lambda_1 \sin^2\theta}{2} \cdot \frac{(L_z - L_3\cos\theta)^2}{\lambda_1^2 \sin^4\theta} + \frac{\lambda_1\dot{\theta}^2}{2} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

$$E = \frac{\lambda_1\dot{\theta}^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$$

(one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2

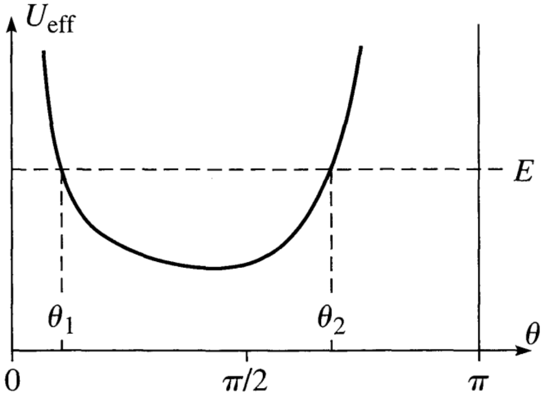
$$E = \frac{\lambda_1 \dot{\theta}^2}{2} + \boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}$$

$U_{eff}(\theta)$

$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$

(one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2



Physics 351 — Friday, April 6, 2018

- Turn in HW10 either today or Monday, as you prefer. Grace will offer an extra help session this Sunday (4/8), 1pm–3pm in DRL 3W2.
- Pick up handout for HW11, due next Friday.
- This weekend, read Taylor's chapter 13 (Hamiltonian mechanics), which we'll start to discuss on Wednesday. Hamiltonians will be the last major topic of the semester.
- We'll spend Monday on coupled oscillators (chapter 11), so if you didn't read it last weekend, please do so now.