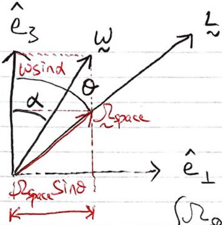
Physics 351 — Monday, April 9, 2018

- Turn in HW10 either today or Wednesday, as you prefer.
- You can see how I solved HW10/q8 for last year's final exam:

  `http://positron.hep.upenn.edu/p351/files/exam2017_solns.pdf`
- HW11 due either Friday or next Monday, as you prefer. One normal-modes problem, one generic Lagrangian problem, three Hamiltonian problems.
- Today, we'll spend one last day to finish Ch10. Then we'll spend one day (Wed) on Ch11 (coupled oscillators). Then finally Friday we'll start Hamiltonians — the last major topic of the semester.
- This space-station video makes it easier to believe that you are seeing the "unstable middle axis" in the Dzhanibekov effect:

`https://commons.wikimedia.org/w/index.php?title=File:Dzhanibekov_effect.ogv`



We know $\mathbf{\Omega}_{space} \parallel \mathbf{L}$

We know $\mathbf{\Omega}_{body} \parallel \pm\hat{e}_3$

$\mathbf{\Omega}_{space}$ is how $\mathbf{\omega}$ precesses wrt $\hat{x}, \hat{y}, \hat{z}$

$\mathbf{\Omega}_{body}$ is how $\mathbf{\omega}$ precesses wrt $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$\mathbf{\omega}$ is how $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate wrt $\hat{x} \hat{y} \hat{z}$

$$\mathbf{\Omega}_{space} = \mathbf{\Omega}_{body} + \mathbf{\omega}$$

$$(\mathbf{\Omega}_{space})_\perp = (\mathbf{\Omega}_{body})_\perp + (\mathbf{\omega})_\perp$$

$$\Omega_{space}\sin\theta = 0 + \omega\sin\alpha \Rightarrow \Omega_{space} = \omega\frac{\sin\alpha}{\sin\theta}$$

$$\mathbf{L} = (\omega_1\hat{e}_1 + \omega_2\hat{e}_2)\lambda_1 + (\omega_3\hat{e}_3)\lambda_3$$

$$\frac{\mathbf{L}}{\lambda_1} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + (\omega_3\hat{e}_3)\frac{\lambda_3}{\lambda_1}$$

$$\left|\left(\frac{\mathbf{L}}{\lambda_1}\right)_\perp\right| = |\omega_1\hat{e}_1 + \omega_2\hat{e}_2| = \omega\sin\alpha = \frac{L}{\lambda_1}\sin\theta$$

$$\Rightarrow \frac{L}{\lambda_1} = \frac{\omega\sin\alpha}{\sin\theta} = \Omega_{space}$$

$$|\mathbf{L}| = \sqrt{\omega_1^2\lambda_1^2 + \omega_2^2\lambda_1^2 + \omega_3^2\lambda_3^2} = \sqrt{(\omega\sin\alpha)^2\lambda_1^2 + (\omega\cos\alpha)^2\lambda_3^2}$$

$$= \omega\sqrt{\lambda_1^2\sin^2\alpha - \lambda_3^2\sin^2\alpha + (\sin^2\alpha + \cos^2\alpha)\lambda_3^2} = \omega\sqrt{(\lambda_1^2 - \lambda_3^2)\sin^2\alpha + \lambda_3^2}$$

Rotate by angle ϕ about $\hat{z}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

x', y' as above; $z' = z$

Rotate by angle θ about $\hat{y}'$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Mnemonic: for infinitessimal rotation angle $\epsilon \ll 1$, $\boldsymbol{r} \to \boldsymbol{r} + \epsilon\hat{\boldsymbol{\omega}} \times \boldsymbol{r}$. So for rotation about $\hat{\boldsymbol{y}}$, $(1,0,0) \to (1,0,-\epsilon)$, since $\epsilon\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\epsilon\hat{\boldsymbol{z}}$.

Euler angles: can move (x, y, z) axes to arbitrary orientation.

Rotate by ϕ about $\hat{z}$

Then rotate by θ about $\hat{y}'$ $(\hat{e}_2')$

Then rotate by ψ about $\hat{z}''$ $(\hat{e}_3)$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \left(\begin{array}{c|c|c} c\theta c\phi c\psi - s\phi s\psi & -c\theta s\phi c\psi - c\phi s\psi & s\theta c\psi \\ c\theta c\phi s\psi + s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\psi \\ -s\theta c\phi & s\theta s\phi & c\theta \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In[2]:= RotationMatrix[ϕ] // MatrixForm

Out[2]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix}$$

In[4]:= RotationMatrix[ϕ, {0, 0, 1}] // MatrixForm

Out[4]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] & 0 \\ \text{Sin}[\phi] & \text{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[5]:= RotationMatrix[θ, {0, 1, 0}] // MatrixForm

Out[5]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix}$$

In[10]:=

```
r1 = RotationMatrix[ϕ, {0, 0, 1}];
r2 = RotationMatrix[θ, {0, 1, 0}];
r3 = RotationMatrix[ψ, {0, 0, 1}];
r3 . r2 . r1 // MatrixForm
```

Out[13]//MatrixForm=

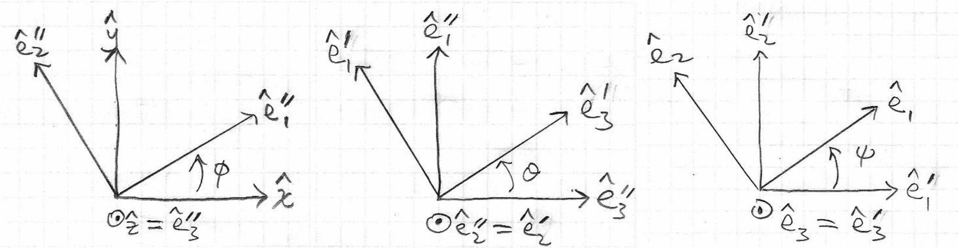
$$\begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Sin}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta] \\ \text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\psi] \\ -\text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\theta] \end{pmatrix}$$

Let the Euler angles ϕ, θ, ψ vary with time, as body rotates.

I'll write out more steps than Taylor does, and I may confuse you by saying $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \rightarrow (\hat{\boldsymbol{e}}_1'', \hat{\boldsymbol{e}}_2'', \hat{\boldsymbol{e}}_3'') \rightarrow (\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3') \rightarrow (\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$.

I do this so that my $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$ are the same as Taylor's.

1. Rotate by ϕ about $\hat{\boldsymbol{z}} \rightarrow \hat{\boldsymbol{e}}_1''$, $\hat{\boldsymbol{e}}_2''$. ($\hat{\boldsymbol{e}}_3'' = \hat{\boldsymbol{z}}$.)
2. Rotate by θ about $\hat{\boldsymbol{e}}_2'' \rightarrow \hat{\boldsymbol{e}}_1'$, $\hat{\boldsymbol{e}}_3'$. ($\hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_2''$.)
3. Rotate by ψ about $\hat{\boldsymbol{e}}_3' \rightarrow \hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$. ($\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_3'$.)



$$\boldsymbol{\omega} \; = \; \dot{\phi}\, \hat{\boldsymbol{z}} + \dot{\theta}\, \hat{\boldsymbol{e}}_2'' + \dot{\psi}\, \hat{\boldsymbol{e}}_3' \; = \; \dot{\phi}\, \hat{\boldsymbol{z}} + \dot{\theta}\, \hat{\boldsymbol{e}}_2' + \dot{\psi}\, \hat{\boldsymbol{e}}_3$$

Remarkable trick: We can write $\boldsymbol{\omega}$ as vector sum of 3 separate angular-velocity vectors, about three successive axes.

Next, project $\boldsymbol{\omega}$ onto more convenient sets of unit vectors.

(Skip this — here for reference)

(orthogonal matrix: inverse = transpose)

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\underset{\sim}{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3' = \dot{\phi}\hat{z} + \dot{\theta}(-\hat{x}\sin\phi + \hat{y}\cos\phi) + \dot{\psi}(\hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta)$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi, +\dot{\theta}\cos\phi, \dot{\phi}) + \dot{\psi}(\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \cos\theta(\hat{z}))$$

$$\underset{\sim}{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi,\ \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,\ \dot{\phi} + \dot{\psi}\cos\theta)$$

(IN SPACE AXES)

$$\left.\begin{aligned} \hat{e}_1'' &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{e}_2'' &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{e}_3'' &= \hat{z} \end{aligned}\right\} \quad \begin{aligned} \hat{x} &= \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi \\ \hat{y} &= \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1' &= \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \\ \hat{e}_3' &= \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \\ \hat{e}_2' &= \hat{e}_2'' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1'' &= \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta \\ \hat{e}_3'' &= -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta \end{aligned}$$

$$\left.\begin{aligned} \hat{e}_1 &= \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \\ \hat{e}_2 &= -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \\ \hat{e}_3 &= \hat{e}_3' \end{aligned}\right\} \quad \begin{aligned} \hat{e}_1' &= \hat{e}_1\cos\psi - \hat{e}_2\sin\psi \\ \hat{e}_2' &= \hat{e}_1\sin\psi + \hat{e}_2\cos\psi \end{aligned}$$

Start from $\boldsymbol{\omega} = \dot{\phi}\,\hat{\boldsymbol{z}} + \dot{\theta}\,\hat{\boldsymbol{e}}_2' + \dot{\psi}\,\hat{\boldsymbol{e}}_3$ and substitute preferred unit vectors.

In the "space" basis [proof on previous page]:

$$\boldsymbol{\omega} = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\hat{\boldsymbol{x}} + (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{\boldsymbol{z}}$$

In the "body" basis [proof on next page]:

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{\boldsymbol{e}}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{\boldsymbol{e}}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3$$

Most convenient for symmetric top ($\lambda_1 = \lambda_2$): in the "primed" basis (i.e. before the final rotation by ψ about $\hat{\boldsymbol{e}}_3$). Note that $\hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_3$.

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' + (\dot{\theta})\hat{\boldsymbol{e}}_2' + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

This last one is easiest to see if you consider the instant at which $\psi = 0$.

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{e}_3'' = \hat{z}$$

$$\hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta$$
$$\hat{e}_2' = \hat{e}_2''$$

$$\hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi$$
$$\hat{e}_3 = \hat{e}_3'$$

$$\hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$

$$\underset{\sim}{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\,\hat{e}_2'' + \dot{\psi}\hat{e}_3'$$
$$= \dot{\phi}\hat{e}_3'' + \dot{\theta}\,\hat{e}_2'' + \dot{\psi}\hat{e}_3'$$
$$= \dot{\phi}\left(-\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta\right) + \dot{\theta}\left(\hat{e}_2'\right) + \dot{\psi}\left(\hat{e}_3'\right)$$
$$= \left[-\dot{\phi}\sin\theta\,(\hat{e}_1') + \dot{\phi}\cos\theta\,(\hat{e}_3')\right] + \dot{\theta}\,(\hat{e}_2') + \dot{\psi}\,(\hat{e}_3')$$
$$= \left[-\dot{\phi}\sin\theta\,(\hat{e}_1\cos\psi - \hat{e}_2\sin\psi) + \dot{\phi}\cos\theta\,\hat{e}_3\right] + \dot{\theta}\,(\hat{e}_1\sin\psi + \hat{e}_2\cos\psi) + \dot{\psi}\,\hat{e}_3'$$
$$= \hat{e}_1\left(-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right) + \hat{e}_2\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + \hat{e}_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$$

Most convenient for **symmetric** top ($\lambda_1 = \lambda_2$):

$$\boldsymbol{\omega} \;=\; (-\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; (\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

This basis makes it easy to write down the top's angular momentum $\boldsymbol{L}$, kinetic energy T, and Lagrangian $\mathcal{L}$.

$$\boldsymbol{L} \;=\; (-\lambda_1\dot{\phi}\sin\theta)\hat{\boldsymbol{e}}_1' \;+\; (\lambda_1\dot{\theta})\hat{\boldsymbol{e}}_2' \;+\; \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\hat{\boldsymbol{e}}_3'$$

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

We then find two ignorable coordinates: ϕ and ψ. So using the corresponding conserved quantities, we can reduce the θ EOM to a single-variable problem.

$$\boldsymbol{\omega} \;=\; (-\dot\phi\sin\theta)\hat{\boldsymbol{e}}'_1 \;+\; (\dot\theta)\hat{\boldsymbol{e}}'_2 \;+\; (\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}'_3$$

$$\boldsymbol{L} \;=\; (-\lambda_1\dot\phi\sin\theta)\hat{\boldsymbol{e}}'_1 \;+\; (\lambda_1\dot\theta)\hat{\boldsymbol{e}}'_2 \;+\; \lambda_3(\dot\phi\cos\theta + \dot\psi)\hat{\boldsymbol{e}}'_3$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot\psi} = \boxed{\lambda_3(\dot\psi + \dot\phi\cos\theta)} \equiv P_\psi \equiv \text{const.} = \lambda_3\omega_3 = \boxed{L_3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = \boxed{\lambda_1\dot\phi\sin^2\theta} + \boxed{\lambda_3(\dot\psi + \dot\phi\cos\theta)}\cos\theta \equiv P_\phi \equiv \text{const.}$$

$$= \boxed{(L_z - L_3\cos\theta)} + \boxed{L_3}\cos\theta = L_z$$

Digression: $\hat{z} = -\hat{e}'_1\sin\theta + \hat{e}'_3\cos\theta$

$$L_z = \underset{\sim}{L}\cdot\hat{z} = \lambda_1\dot\phi\sin^2\theta + \boxed{\lambda_3(\dot\psi + \dot\phi\cos\theta)}\cos\theta$$

$$L_z = \lambda_1\dot\phi\sin^2\theta + L_3\cos\theta$$

$$\Rightarrow \lambda_1\dot\phi\sin^2\theta = L_z - L_3\cos\theta$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

θ equation of motion: (derivation 2 pages ahead)

$$\lambda_1 \ddot\theta = \lambda_1 \boxed{\dot\phi^2} \sin\theta\cos\theta - (\lambda_3\omega_3)\boxed{\dot\phi}\sin\theta + MgR\sin\theta$$

First consider case where θ = constant. Since $\lambda_1 \dot\phi \sin^2\theta = L_z - L_3\cos\theta$, $\theta = \text{const} \Rightarrow \dot\phi = \text{const} \equiv \Omega$

$$0 = \lambda_1 \boxed{\Omega^2} \cancel{\sin\theta}\cos\theta - \lambda_3\omega_3\boxed{\Omega}\cancel{\sin\theta} + MgR\cancel{\sin\theta}$$

$$(\lambda_1\cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR = 0$$

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1\cos\theta MgR}}{2\lambda_1\cos\theta}$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1\cos\theta MgR}{(\lambda_3\omega_3)^2}}\right)$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1\cos\theta\, MgR}{(\lambda_3\omega_3)^2}}\right)$$

has 2 real solutions if $(\lambda_3\omega_3)^2 > 4\lambda_1 MgR\cos\theta$
("if ω_3 is large enough")

Math is simplest if $(\lambda_3\omega_3)^2 \gg 4\lambda_1 MgR\cos\theta$
("ω_3 is very large")

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \left(1 - \frac{2\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2}\right)\right)$$

$$\Omega_+ \simeq \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} \qquad \Omega_- \simeq \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\cdot\frac{2\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2} = \boxed{\frac{MgR}{\lambda_3\omega_3} = \Omega_-}$$

This is Ω_{space} for tossed football with no torque

(See HW)

This can be seen from

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$MgR\sin\theta = \lambda_3\omega_3\,\Omega\sin\theta$$

(Skip: Just in case you wanted to see the θ EOM derived.)

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\phi\cos\theta + \dot\psi)^2 - MgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \lambda_1 \dot\phi^2 \sin\theta\cos\theta + \lambda_3(\dot\phi\cos\theta + \dot\psi)(-\dot\phi\sin\theta) + MgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{d}{dt}(\lambda_1\dot\theta) = \lambda_1\ddot\theta$$

$$\lambda_1\ddot\theta = \lambda_1\dot\phi^2\sin\theta\cos\theta - \lambda_3(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta + MgR\sin\theta$$

$$\lambda_1\ddot\theta = \lambda_1\dot\phi^2\sin\theta\cos\theta - \lambda_3\omega_3\dot\phi\sin\theta + MgR\sin\theta$$

$$\lambda_1 \dot{\phi} \sin^2\theta = L_z - L_3\cos\theta \Rightarrow \dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$$

$$E = T + U = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + MgR\cos\theta$$

$$E = \frac{1}{2}\lambda_1 \sin^2\theta\, \dot{\phi}^2 + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3 \omega_3^2 + MgR\cos\theta$$

$$E = \frac{\lambda_1\sin^2\theta}{2}\cdot\frac{(L_z - L_3\cos\theta)^2}{\lambda_1^2\sin^4\theta} + \frac{\lambda_1\dot{\theta}^2}{2} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

$$E = \frac{\lambda_1\dot{\theta}^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$$ (one dimensional problem)

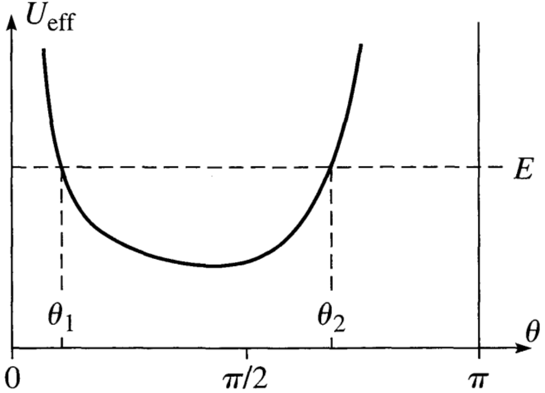
θ "nutates" back and forth between θ_1 and θ_2

$$E = \frac{\lambda_1 \dot{\theta}^2}{2} + \boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}$$

$U_{eff}(\theta)$

$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$ (one dimensional problem)

θ "nutates" back and forth between θ_1 and θ_2



(Taylor 10.35) A rigid body consists of:

$$\begin{aligned} m \text{ at } (a,0,0) &= a(1,\ 0,\ \ 0) \\ 2m \text{ at } (0,a,a) &= a(0,\ 1,\ \ 1) \\ 3m \text{ at } (0,a,-a) &= a(0,\ 1,-1) \end{aligned}$$

Find inertia tensor $\underline{\underline{\boldsymbol{I}}}$, its principal moments, and the principal axes.

$$I_{xx} = \sum m(y^2+z^2) = ma^2(2\cdot 2+2\cdot 3) = 10ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{xy} = -\sum mxy = -ma^2(0) = 0$$

$$I_{xz} = -\sum mxz = -ma^2(0) = 0$$

$$I_{yz} = -\sum myz = -ma^2(2-3) = ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} ma^2$$

$$\underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda\,\underset{\sim}{\omega} \Rightarrow \left(\underline{\underline{I}} - \lambda \mathbb{1}\right)\underset{\sim}{\omega} = 0$$

$$\Rightarrow \det\left(\underline{\underline{I}} - \lambda \mathbb{1}\right) = 0$$

$0 = (10-\lambda)(6-\lambda)^2 - (10-\lambda) \Rightarrow \lambda = 10$ or $(6-\lambda)^2 = 1$

$6-\lambda = 1 \Rightarrow \lambda = 5$, $\quad 6-\lambda = -1 \Rightarrow \lambda = 7 \qquad \lambda \in \{10, 7, 5\}$

$$\lambda \in \{10, 7, 5\}\, ma^2$$

$$0 = \begin{pmatrix} 10-10 & 0 & 0 \\ 0 & 6-10 & 1 \\ 0 & 1 & 6-10 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4y+z \\ y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = z_1 = 0 \Rightarrow \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-7 & 0 & 0 \\ 0 & 6-7 & 1 \\ 0 & 1 & 6-7 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3x \\ z-y \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0,\ y_2 = z_2 \Rightarrow \hat{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-5 & 0 & 0 \\ 0 & 6-5 & 1 \\ 0 & 1 & 6-5 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5x \\ y+z \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3 = 0,\ y_3 = -z_3 \Rightarrow \hat{e}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

eigenvectors {{10,0,0},{0,6,1},{0,1,6}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}\right]$$

Results:

$v_1 = (1, 0, 0)$

$v_2 = (0, 1, 1)$

$v_3 = (0, -1, 1)$

Corresponding eigenvalues:

$\lambda_1 = 10$

$\lambda_2 = 7$

$\lambda_3 = 5$

```
In[1]:= m = {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

Out[1]= {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

In[2]:= MatrixForm[m]
```

2]//MatrixForm=

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

```
In[3]:= Eigenvalues[m]

Out[3]= {10, 7, 5}

In[4]:= Eigenvectors[m]

Out[4]= {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}

In[5]:= Eigensystem[m]

Out[5]= {{10, 7, 5},
          {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}}
```

Physics 351 — Monday, April 9, 2018

- Turn in HW10 either today or Wednesday, as you prefer.
- You can see how I solved HW10/q8 for last year's final exam:

  `http://positron.hep.upenn.edu/p351/files/exam2017_solns.pdf`
- HW11 due either Friday or next Monday, as you prefer. One normal-modes problem, one generic Lagrangian problem, three Hamiltonian problems.
- Today, we'll spend one last day to finish Ch10. Then we'll spend one day (Wed) on Ch11 (coupled oscillators). Then finally Friday we'll start Hamiltonians — the last major topic of the semester.
- This space-station video makes it easier to believe that you are seeing the "unstable middle axis" in the Dzhanibekov effect:

`https://commons.wikimedia.org/w/index.php?title=File:Dzhanibekov_effect.ogv`