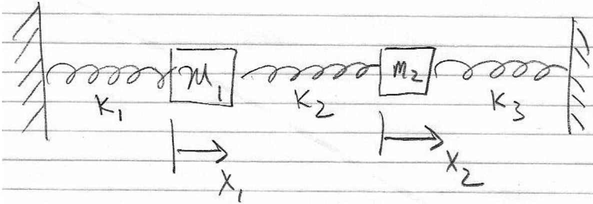
Physics 351 — Wednesday, April 11, 2018

- Turn in HW10 today if you have not already done so.
- HW11 due either Friday or next Monday, as you prefer. One normal-modes problem, one generic Lagrangian problem, three Hamiltonian problems.
- HW help this week: Grace is in DRL 2C2 Thu 5:30–8:30pm, as usual. **Bill will be in DRL 3W2 Sunday (4/15) 2–5pm, instead of today's usual Wed afternoon.**
- We'll spend today (only) on Ch11 (coupled oscillators). Then finally Friday we'll start Hamiltonians — the last major topic of the semester.
- A fun space-station video using a box of playing cards to illustrate Euler's equations: `https://youtu.be/fPI-rSwAQNg`



$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \ddot{x}_2 = k_2 (x_1 - x_2) - k_3 x_2$$

$$m_1 \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$$

$$m_2 \ddot{x}_2 = k_2 x_1 - (k_2 + k_3) x_2$$

$$\ddot{x}_1 = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\ddot{x}_2 = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$

$$\ddot{X}_1 = -\frac{k_1+k_2}{m_1} X_1 + \frac{k_2}{m_1} X_2$$

$$\ddot{X}_2 = \frac{k_2}{m_2} X_1 - \frac{k_2+k_3}{m_2} X_2$$

$$\begin{pmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{pmatrix} = -\begin{pmatrix} +\frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & +\frac{k_2+k_3}{m_2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\underset{\sim}{\ddot{X}} = -\left(\underline{\underline{M^{-1}K}}\right) \underset{\sim}{X}$$

Plug in $\underset{\sim}{X} = \underset{\sim}{A}\, e^{i\omega t} \Rightarrow -\omega^2 \underset{\sim}{A} = -\left(\underline{\underline{M^{-1}K}}\right)\underset{\sim}{A}$

$\left(\underline{\underline{M^{-1}K}} - \omega^2\right) \underset{\sim}{A} = 0$ eigenvalue equation

In[15]:=
```
ClearAll["Global`*"];
MinverseK := {{(k1 + k2) / m1, -k2 / m1},
   {-k2 / m2, (k2 + k3) / m2}};
MatrixForm[MinverseK]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} \frac{k1+k2}{m1} & -\frac{k2}{m1} \\ -\frac{k2}{m2} & \frac{k2+k3}{m2} \end{pmatrix}$$

In[18]:=
```
Eigenvalues[MinverseK]
```

Out[18]=

$$\left\{ \frac{1}{2\ m1\ m2} \left(k2\ m1 + k3\ m1 + k1\ m2 + k2\ m2 - \sqrt{\left((-k2\ m1 - k3\ m1 - k1\ m2 - k2\ m2)^2 - 4\ (k1\ k2\ m1\ m2 + k1\ k3\ m1\ m2 + k2\ k3\ m1\ m2) \right)} \right), \frac{1}{2\ m1\ m2} \left(k2\ m1 + k3\ m1 + k1\ m2 + k2\ m2 + \sqrt{\left((-k2\ m1 - k3\ m1 - k1\ m2 - k2\ m2)^2 - 4\ (k1\ k2\ m1\ m2 + k1\ k3\ m1\ m2 + k2\ k3\ m1\ m2) \right)} \right) \right\}$$

In[19]:=

```
Eigenvectors[MinverseK]
```

Out[19]=

$$\left\{\left\{-\frac{1}{2\,k2\,m1}\left(-k2\,m1-k3\,m1+k1\,m2+k2\,m2-\sqrt{\left((-k2\,m1-k3\,m1-k1\,m2-k2\,m2)^2-4\,(k1\,k2\,m1\,m2+k1\,k3\,m1\,m2+k2\,k3\,m1\,m2)\right)}\right),\ 1\right\},\right.$$

$$\left.\left\{-\frac{1}{2\,k2\,m1}\left(-k2\,m1-k3\,m1+k1\,m2+k2\,m2+\sqrt{\left((-k2\,m1-k3\,m1-k1\,m2-k2\,m2)^2-4\,(k1\,k2\,m1\,m2+k1\,k3\,m1\,m2+k2\,k3\,m1\,m2)\right)}\right),\ 1\right\}\right\}$$

```
]:= matrix =
      MinverseK /. {k1 → k, k3 → k, k2 → k, m1 → m, m2 → m};
    MatrixForm[matrix]
```

MatrixForm=

$$\begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix}$$

```
]:= Eigenvalues[matrix]
```

3]= $\left\{\frac{3k}{m}, \frac{k}{m}\right\}$

```
]:= Eigenvectors[matrix]
```

4]= {{-1, 1}, {1, 1}}

9]:= matrix =
  MinverseK /. {k1 → k, k3 → k, k2 → k, m1 → m, m2 → 2 m};
MatrixForm[matrix]

/MatrixForm=

$$\begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{2m} & \frac{k}{m} \end{pmatrix}$$

1]:= Eigenvalues[matrix]

1]= $\left\{\frac{\left(3+\sqrt{3}\right) k}{2 m}, \frac{\left(3-\sqrt{3}\right) k}{2 m}\right\}$

2]:= Eigenvectors[matrix]

2]= $\left\{\left\{-1-\sqrt{3}, 1\right\}, \left\{-1+\sqrt{3}, 1\right\}\right\}$

3]:= N[%]

3]= {{-2.73205, 1.}, {0.732051, 1.}}

4]:= N[Eigenvalues[matrix]]

4]= $\left\{\frac{2.36603 k}{m}, \frac{0.633975 k}{m}\right\}$

5]:= %[[1]] / %[[2]]

5]= 3.73205

25]:=

```
matrix = MinverseK /. {k1 → k, k3 → k, m1 → m, m2 → m};
MatrixForm[matrix]
```

]//MatrixForm=

$$\begin{pmatrix} \frac{k+k2}{m} & -\frac{k2}{m} \\ -\frac{k2}{m} & \frac{k+k2}{m} \end{pmatrix}$$

27]:=

```
Eigenvalues[matrix]
```

[27]= $\left\{\frac{k}{m}, \frac{k+2\,k2}{m}\right\}$

28]:=

```
Eigenvectors[matrix]
```

[28]= {{1, 1}, {-1, 1}}

Our two normal modes are

① $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 e^{i(\omega_1 t - \delta_1)}$ with $\omega_1 = \sqrt{\frac{k}{m}}$

② $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} A_2 e^{i(\omega_2 t - \delta_2)}$ with $\omega_2 = \sqrt{\frac{k + 2k_2}{m}}$

Suppose at $t=0$ we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix}$
which is a superposition of the two modes:

$$\begin{pmatrix} A \\ 0 \end{pmatrix} = \frac{A}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{A}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow A_1 = \tfrac{1}{2}A \quad A_2 = -\tfrac{1}{2}A$$

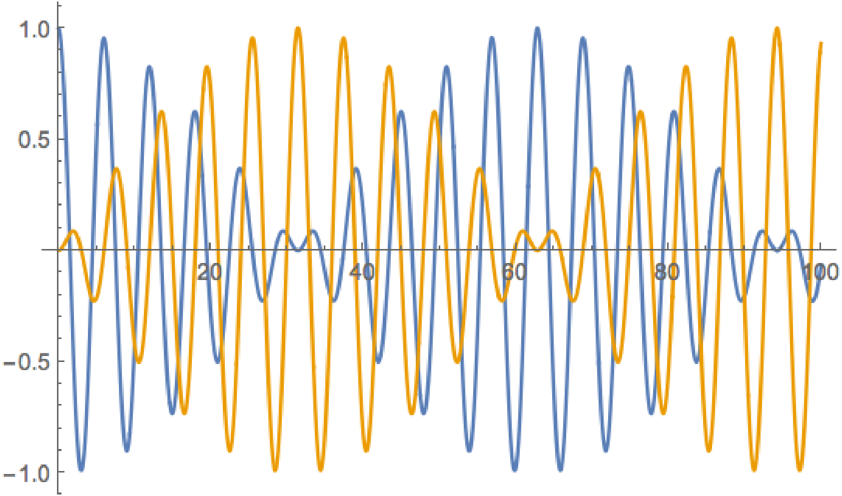
$$\delta_1 = 0 \qquad \delta_2 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A \\ A \end{pmatrix} e^{i\omega_1 t} - \frac{1}{2}\begin{pmatrix} -A \\ A \end{pmatrix} e^{i\omega_2 t}$$ (take real part for physical motion)

$$x_1(t) = \frac{A}{2}\left(\cos\omega_1 t + \cos\omega_2 t\right)$$

$$x_2(t) = \frac{A}{2}\left(\cos\omega_1 t - \cos\omega_2 t\right)$$

```
x1[t_] := (A / 2) (Cos[ω1 t] + Cos[ω2 t]);
x2[t_] := (A / 2) (Cos[ω1 t] - Cos[ω2 t]);
ω1 = 1.0 ; ω2 = 1.1; A = 1;
Plot[{x1[t], x2[t]}, {t, 0, 100}]
```

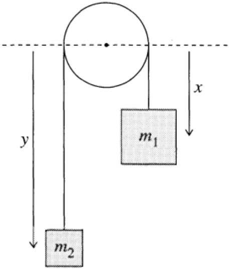


Let's try out Taylor's "procedure" for Hamilton's equations.

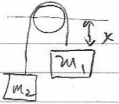
This example illustrates the general procedure to be followed in setting up Hamilton's equations for any given system:

1. Choose suitable generalized coordinates, $q_1, \cdots, q_n$.
2. Write down the kinetic and potential energies, T and U, in terms of the q's and $\dot{q}$'s.
3. Find the generalized momenta $p_1, \cdots, p_n$. (We are now assuming our system is conservative, so U is independent of $\dot{q}_i$ and we can use $p_i = \partial T/\partial \dot{q}_i$. In general, one must use $p_i = \partial \mathcal{L}/\partial \dot{q}_i$.)
4. Solve for the $\dot{q}$'s in terms of the p's and q's.
5. Write down the Hamiltonian $\mathcal{H}$ as a function of the p's and q's. [Provided our coordinates are "natural" (relation between generalized coordinates and underlying Cartesians is independent of time), $\mathcal{H}$ is just the total energy $\mathcal{H} = T + U$, but when in doubt, use $\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$. See Problems 13.11 and 13.12.]
6. Write down Hamilton's equations (13.25).

Taylor 13.3. Consider the Atwood machine of Figure 13.2, but suppose that the pulley is a uniform disc of mass M and radius R. Using x as your generalized coordinate, write down $\mathcal{L}$, the generalized momentum p, and $\mathcal{H} = p\dot{x} - \mathcal{L}$. Write Hamilton's equations and use them to find $\ddot{x}$.



Taylor 13.3



$$U = (m_2 - m_1) g x$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2$$
$$= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2$$

$$\mathcal{L} = T - U = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\dot{x}^2 + (m_1 - m_2) g x$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{M}{2}\right)\dot{x} \implies \dot{x} = \frac{p_x}{m_1 + m_2 + \frac{M}{2}}$$

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p_x^2}{m_1 + m_2 + \frac{M}{2}} - \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\left(\frac{p_x}{m_1 + m_2 + \frac{M}{2}}\right)^2 + (m_2 - m_1) g x$$

$$\mathcal{H} = \frac{p_x^2}{2\left(m_1 + m_2 + \frac{M}{2}\right)} + (m_2 - m_1) g x$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m_1 + m_2 + \frac{M}{2}} \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = (m_1 - m_2) g$$

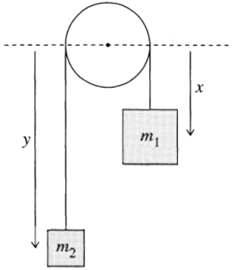
$$\ddot{x} = \frac{\dot{p}_x}{m_1 + m_2 + \frac{M}{2}} = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{M}{2}}$$

By the way, if you take the three original Cartesian coordinates to be x, y, and ϕ, then the one generalized coordinate is $q = x$.

$$x = q$$

$$y = \text{const.} - q$$

$$\phi = q/R$$



All of these are time-independent and don't involve the velocities, so the generalized coordinate q is "natural" (or "scleronomous" in Goldstein's language). Goldstein's word for "unnatural" is "rheonomous."

So we found

$$\mathcal{H} = T + U$$

Taylor 13.11. The simple form $\mathcal{H} = T + U$ is true only if your generalized coordinates are "natural" (relation between generalized and underlying Cartesian coordinates is independent of time). If the generalized coordinates are not "natural," you must use

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L}$$

To illustrate: Two children play catch inside a railroad car moving with varying speed V along a straight horizontal track. For generalized coordinates you can use (x, y, z) of the ball relative to a fixed point in the car, but in setting up $\mathcal{H}$ you must use coordinates in an inertial frame. Find $\mathcal{H}$ for the ball and show that it is not equal to $T + U$ (neither as measured in the car, nor as measured in the ground-based frame).

Taylor wrote way back on p. 270 (Eq. 7.91) that $\mathcal{H} = T + U$ if

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1, \cdots, q_n)$$

(i.e. there is no "t" and no $\dot{q}_i$ when writing $\boldsymbol{r}_\alpha$ in terms of the q_i's.)

$$T = \frac{1}{2}m\left((\dot{x}+V)^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = mgz$$

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+V) \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L} = p_x\left(\frac{p_x}{m} - V\right) + p_y\left(\frac{p_y}{m}\right) + p_z\left(\frac{p_z}{m}\right)$$
$$- \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + mgz$$

$$\mathcal{H} = \frac{p_x^2}{2m} - p_x V + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz$$

$$(T+U)_{\substack{\text{train}\\\text{car}\\\text{frame}}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = \frac{1}{2}m\left(\left(\frac{p_x}{m} - V\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + mgz$$
$$= \frac{\vec{p}^{\,2}}{2m} - p_x V + \frac{1}{2}mV^2 + mgz \neq \mathcal{H}$$

$$(T+U)_{\substack{\text{ground}\\\text{frame}}} = \frac{1}{2}m\left((\dot{x}+V)^2 + \dot{y}^2 + \dot{z}^2\right) + mgz$$
$$= \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + mgz = \frac{\vec{p}^{\,2}}{2m} + mgz \neq \mathcal{H}$$

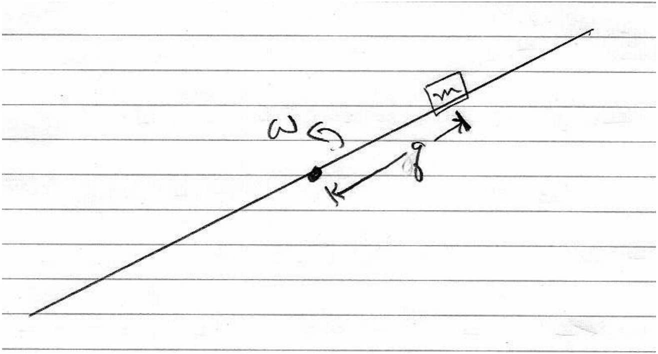
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} - V \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0 \implies \ddot{x} = -\dot{V}$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \qquad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0 \implies \ddot{y} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -mg \implies \ddot{z} = -g$$

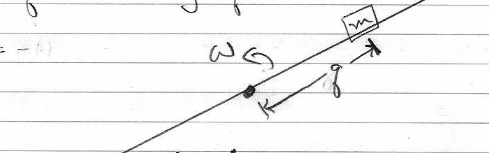
Taylor 13.12. Same as previous problem but use this system:

A bead of mass m is threaded on a frictionless, straight rod, which lies in a horizontal plane and is forced to spin with constant angular velocity ω about a vertical axis through the midpoint of the rod. Find $\mathcal{H}$ for the bead and show that $\mathcal{H} \neq T + U$.



(I suggest this generalized coordinate q.)

$x = q\cos\omega t \qquad y = q\sin\omega t$



$$\dot{x} = \dot{q}\cos\omega t - \omega q\sin\omega t$$
$$\dot{y} = \dot{q}\sin\omega t + \omega q\cos\omega t$$

$$T = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \quad \longrightarrow \mathcal{L} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \implies \dot{q} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p\left(\frac{p}{m}\right) - \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 - \tfrac{1}{2}m\omega^2 q^2 = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 q^2$$

$$(T+U)_{\substack{\text{relative}\\\text{to rod}}} = \tfrac{1}{2}m\dot{q}^2 = \frac{p^2}{2m} \neq \mathcal{H}$$

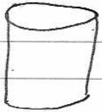
$$(T+U)_{\substack{\text{relative}\\\text{to}\\\text{ground}}} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \neq \mathcal{H}$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = +m\omega^2 q \implies \ddot{q} = \omega^2 q$$

(q increases exponentially)

Taylor 13.13. Consider a particle of mass m constrained to move on a frictionless cylinder or radius R, given by the equation $\rho = R$ in (ρ, ϕ, z) coords. The mass is subject to force $\boldsymbol{F} = -kr\hat{\boldsymbol{r}}$, where k is a positive constant, r is distance from the origin, and $\hat{\boldsymbol{r}}$ points away from the origin. Using z and ϕ as generalized coordinates, find $\mathcal{H}$, write down Hamilton's equations, and describe the motion.

$$\vec{F} = -kr\hat{r} \Rightarrow U = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(\rho^2+z^2)$$
$$= \tfrac{1}{2}k(R^2+z^2)$$



so might as well write $U = \frac{1}{2}kz^2$

$$T = \tfrac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2)$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$$

$$\mathcal{H} = T+U = \frac{p_z^2}{2m} + \tfrac{1}{2}mR^2\left(\frac{p_\phi}{mR^2}\right)^2 + \tfrac{1}{2}kz^2 = \frac{p_z^2}{2m} + \frac{p_\phi^2}{2mR^2} + \tfrac{1}{2}kz^2$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \Rightarrow \ddot{z} = -\frac{k}{m}z$$

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