Physics 351 — Monday, April 16, 2018

- Turn in HW11 today. HW12, due either Friday 4/20 or Monday 4/23, as you prefer.
- Two optional/XC readings: ch14 (collisions/scattering [easy]) and ch16 (continuum mechanics [long]). If you answer the questions (click 4/18 and 4/27, respectively, on the reading-response page), you can get +10 for ch14 and +20 for ch16 added into your reading total, which doesn't count against your maximum +5% overall XC boost.
- You can also, if you wish, earn ordinary XC by solving any problems you like (even easy ones) from ch14 or ch16, and you can solve any ** or *** problems you like from ch11.
- Our topic for the rest of the semester is Hamiltonian mechanics, but you'll read a few supplementary things for enrichment. Final exam will cover ch 7,9,10,13 [only].
- Does anyone object to a future service dog coming to class this Friday? (He'll sit near the front of the room, I think.)

(This was the last thing we did on Friday.)

Taylor 13.11. The simple form $\mathcal{H} = T + U$ is true only if your generalized coordinates are "natural" (relation between generalized and underlying Cartesian coordinates is independent of time). If the generalized coordinates are not "natural," you must use

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L}$$

To illustrate: Two children play catch inside a railroad car moving with varying speed V along a straight horizontal track. For generalized coordinates you can use (x, y, z) of the ball relative to a fixed point in the car, but in setting up $\mathcal{H}$ you must use coordinates in an inertial frame. Find $\mathcal{H}$ for the ball and show that it is not equal to $T + U$ (neither as measured in the car, nor as measured in the ground-based frame).

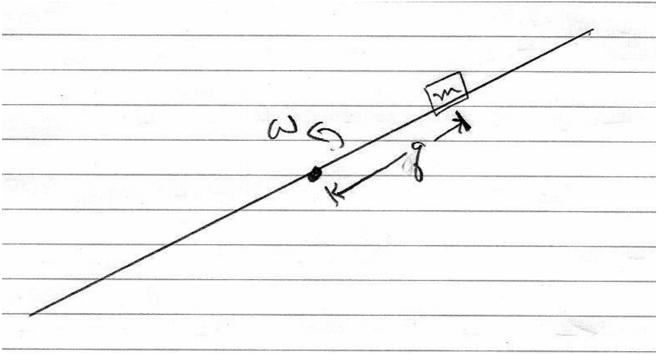
Taylor wrote way back on p. 270 (Eq. 7.91) that $\mathcal{H} = T + U$ if

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1, \cdots, q_n)$$

(i.e. there is no "t" and no $\dot{q}_i$ when writing $\boldsymbol{r}_\alpha$ in terms of the q_i's.)

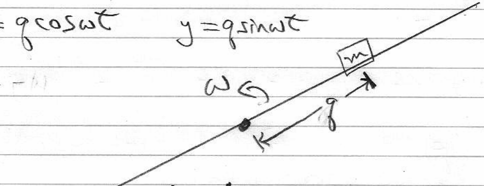
Taylor 13.12. Same as previous problem but use this system:

A bead of mass m is threaded on a frictionless, straight rod, which lies in a horizontal plane and is forced to spin with constant angular velocity ω about a vertical axis through the midpoint of the rod. Find $\mathcal{H}$ for the bead and show that $\mathcal{H} \neq T + U$.



(I suggest this generalized coordinate q.)

$x = q\cos\omega t \qquad y = q\sin\omega t$



$$\dot{x} = \dot{q}\cos\omega t - \omega q\sin\omega t$$
$$\dot{y} = \dot{q}\sin\omega t + \omega q\cos\omega t$$

$$T = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \quad \rightarrow \mathcal{L} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2$$

$$p = \frac{\partial\mathcal{L}}{\partial\dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p\left(\frac{p}{m}\right) - \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 - \tfrac{1}{2}m\omega^2 q^2 = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 q^2$$

$$(T+U)_{\substack{\text{relative}\\ \text{to rod}}} = \tfrac{1}{2}m\dot{q}^2 = \frac{p^2}{2m} \neq \mathcal{H}$$

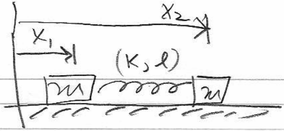
$$(T+U)_{\substack{\text{relative}\\ \text{to}\\ \text{ground}}} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \neq \mathcal{H}$$

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q} = +m\omega^2 q \quad \Rightarrow \ddot{q} = \omega^2 q$$

(q increases exponentially)

Morin 15.28. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let their positions be x_1 and x_2. Find $\mathcal{H}$ in terms of x_1 and x_2 and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.28



$$T = \tfrac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) \qquad U = \tfrac{1}{2} k (x_1 + \ell - x_2)^2$$

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial T}{\partial \dot{x}_1} = m \dot{x}_1 \qquad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial T}{\partial \dot{x}_2} = m \dot{x}_2$$

$$\dot{x}_1 = \frac{p_1}{m} \qquad \dot{x}_2 = \frac{p_2}{m} \qquad \Rightarrow \tfrac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) = \frac{p_1^2 + p_2^2}{2m}$$

$$\mathcal{H} = p_1 \dot{x}_1 + p_2 \dot{x}_2 - \mathcal{L} = \frac{p_1^2 + p_2^2}{m} - \left(\frac{p_1^2 + p_2^2}{2m} - \tfrac{1}{2} k (x_1 + \ell - x_2)^2 \right)$$

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \tfrac{1}{2} k (x_1 + \ell - x_2)^2 = T + U$$

(expected result, as coords are "natural")

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{p_1}{m} \qquad \dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m}$$

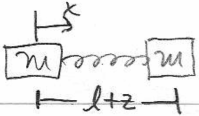
$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = -k (x_1 + \ell - x_2) \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = +k (x_1 + \ell - x_2)$$

$$\Rightarrow \ddot{x}_1 = -\frac{k}{m} (x_1 + \ell - x_2)$$

$$\ddot{x}_2 = \frac{k}{m} (x_1 + \ell - x_2)$$

Morin 15.8. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let the position of the left bead be x, and let z be the stretch of the spring (w.r.t. equilibrium). Find $\mathcal{H}$ in terms of x and z and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.8



$$U = \frac{1}{2} k z^2 \qquad T = \frac{m}{2}\left(\dot{x}^2 + (\dot{x}+\dot{z})^2\right)$$
$$= \frac{1}{2} m \left(2\dot{x}^2 + 2\dot{x}\dot{z} + \dot{z}^2\right)$$
$$= m\dot{x}^2 + m\dot{x}\dot{z} + \frac{1}{2} m \dot{z}^2$$

In terms of last problem's coordinates: $\left.\begin{aligned} x_1 &= x \\ x_2 &= x + z + l \end{aligned}\right\} \begin{aligned} &\neq f(t) \\ &\neq f(\dot{x}) \\ &\neq f(\dot{z}) \end{aligned}$ ✓

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2m\dot{x} + m\dot{z}$$

$$P_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{x} + m\dot{z} \longrightarrow \dot{x} + \dot{z} = \frac{P_z}{m}$$

$$P_x - P_z = m\dot{x} \Rightarrow \dot{x} = \frac{P_x - P_z}{m}$$

$$m\dot{z} = P_z - m\dot{x} = P_z - (P_x - P_z) = 2P_z - P_x \Rightarrow \dot{z} = \frac{2P_z - P_x}{m}$$

$$\mathcal{H} = \frac{m}{2}\left(\left(\frac{P_x - P_z}{m}\right)^2 + \left(\frac{P_z}{m}\right)^2\right) + \frac{kz^2}{2} = \frac{1}{2m}\left(P_x^2 - 2P_xP_z + 2P_z^2\right) + \frac{kz^2}{2}$$

$$\mathcal{H} = \frac{P_x^2}{2m} - \frac{P_xP_z}{m} + \frac{P_z^2}{m} + \frac{kz^2}{2}$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x} = \frac{P_x}{m} - \frac{P_z}{m} \qquad \dot{z} = \frac{\partial \mathcal{H}}{\partial P_z} = \frac{2P_z}{m} - \frac{P_x}{m}$$

$$\dot{P}_x = 0 \qquad \dot{P}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz$$

$$\left(\ddot{x} = +\frac{k}{m} z = -\frac{\ddot{z}}{2}\right) \qquad \ddot{z} = \frac{2}{m}(-kz) = -\frac{2k}{m} z$$

What do you expect the general solution to the motion to look like? (It's similar to HW problem 2 you just solved, the pendulum mounted on a frictionless horizontal rail.)

Let's try same problem one more way:

let X = cm position $= \frac{1}{2}(x_1 + x_2)$

let $z + l = x_2 - x_1$

$$\Rightarrow T = \frac{1}{2}\underbrace{(2m)}_{M_{total}}\dot{X}^2 + \frac{1}{2}\underbrace{\left(\frac{m}{2}\right)}_{\mu = \text{reduced mass}}\dot{z}^2 \quad , \quad U = \frac{1}{2}kz^2$$

$$T = m\dot{X}^2 + \frac{1}{4}m\dot{z}^2 \qquad P_X = 2m\dot{X} \qquad P_z = \frac{1}{2}m\dot{z}$$

coords "natural" $\Rightarrow$ $\quad \dot{X} = \frac{P_X}{2m} \qquad \dot{z} = \frac{2P_z}{m}$

$$\mathcal{H} = T + U = m\left(\frac{P_X}{2m}\right)^2 + \frac{m}{4}\left(\frac{2P_z}{m}\right)^2 + \frac{1}{2}kz^2$$

$$\mathcal{H} = \frac{P_X^2}{4m} + \frac{P_z^2}{m} + \frac{1}{2}kz^2$$

$$\dot{P}_X = -\frac{\partial \mathcal{H}}{\partial X} = 0 \quad , \quad \dot{X} = \frac{\partial \mathcal{H}}{\partial P_X} = \frac{P_X}{2m} \Rightarrow \ddot{X} = 0$$

cm velocity constant

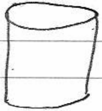
$$\dot{P}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \qquad \dot{z} = \frac{\partial \mathcal{H}}{\partial P_z} = \frac{2P_z}{m}$$

$$\Rightarrow \ddot{z} = -\frac{2k}{m}z = -\frac{k}{\mu}z \rightarrow$$

relative coordinate oscillates sinusoidally

Taylor 13.13. Consider a particle of mass m constrained to move on a frictionless cylinder or radius R, given by the equation $\rho = R$ in (ρ, ϕ, z) coords. The mass is subject to force $\boldsymbol{F} = -kr\hat{\boldsymbol{r}}$, where k is a positive constant, r is distance from the origin, and $\hat{\boldsymbol{r}}$ points away from the origin. Using z and ϕ as generalized coordinates, find $\mathcal{H}$, write down Hamilton's equations, and describe the motion.

$$\vec{F} = -kr\hat{r} \Rightarrow \mathcal{U} = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(\rho^2+z^2)$$
$$= \tfrac{1}{2}k(R^2+z^2)$$



so might as well write $\mathcal{U} = \tfrac{1}{2}kz^2$

$$T = \tfrac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2)$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$$

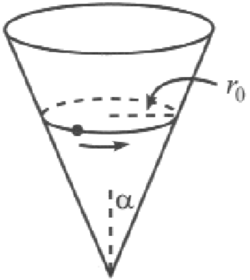
$$\mathcal{H} = T + \mathcal{U} = \frac{p_z^2}{2m} + \tfrac{1}{2}mR^2\left(\frac{p_\phi}{mR^2}\right)^2 + \tfrac{1}{2}kz^2 = \frac{p_z^2}{2m} + \frac{p_\phi^2}{2mR^2} + \tfrac{1}{2}kz^2$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \Rightarrow \ddot{z} = -\frac{k}{m}z$$

Here's a familiar problem from HW5. Let's work through it it using Hamilton's equations instead.

3. A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \qquad U = mgz$$

$$\rho = z\tan\alpha \Rightarrow z = \rho\cot\alpha \equiv c\rho$$

Generalized coordinates : ρ, ϕ ("natural")

$$\left[z = c\rho \qquad x = \rho\cos\phi \qquad y = \rho\sin\phi\right]$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + c^2\dot{\rho}^2\right) \qquad U = mgc\rho$$

$$\mathcal{H} = T + U = \frac{m}{2}\left((1+c^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mgc\rho$$

need to rewrite in terms of p_ρ and p_ϕ

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\rho^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{m\rho^2}$$

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = m(1+c^2)\dot{\rho} \Rightarrow \dot{\rho} = \frac{p_\rho}{m(1+c^2)}$$

$$\mathcal{H} = \frac{m}{2}\left((1+c^2)\left(\frac{p_\rho}{m(1+c^2)}\right)^2 + \rho^2\left(\frac{p_\phi}{m\rho^2}\right)^2\right) + mgc\rho$$

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

$p_\phi \equiv \text{const.}$ (ϕ is ignorable/ cyclic)

$\longrightarrow$ we have a 1D problem

$$\dot{p}_\rho = \frac{\partial \mathcal{H}}{\partial \rho} = -\frac{p_\phi^2}{m\rho^3} + mgc \qquad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial p_\rho} = \frac{-p_\rho}{m(1+c^2)}$$

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right)$$

$$\ddot{\rho} = 0 \Rightarrow mgc = \frac{p_\phi^2}{mr_0^3} \Rightarrow \boxed{r_0^3 = \frac{p_\phi^2}{m^2gc}} = \frac{(mr_0^2\omega_0)^2}{m^2gc} \Rightarrow \omega = \sqrt{\frac{gc}{r_0}}$$

Consider small oscillations of ρ about r_0.

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{m\rho^3} - mgc\right) \equiv f(\rho)$$

$$f(r_0+\varepsilon) = f(r_0) + \varepsilon f'(r_0) + \cancel{\mathcal{O}(\varepsilon^2)}$$

$$f(r_0) = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{m r_0^3} - mgc\right) = 0$$

$$f'(\rho) = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{m}\right)\left(\frac{-3}{\rho^4}\right) = -\frac{3P_\phi^2}{m^2(1+c^2)\rho^4}$$

$$f'(r_0) = -\frac{3P_\phi^2}{m^2(1+c^2)r_0^4}$$

$$\rho = r_0 + \varepsilon \Rightarrow \ddot{\varepsilon} = \ddot{\rho}$$

$$\ddot{\varepsilon} = -\frac{3P_\phi^2}{m^2(1+c^2)r_0^4}\,\varepsilon \quad \Rightarrow \Omega^2 = \frac{3P_\phi^2}{m^2(1+c^2)r_0^4} = \frac{3mgc}{mr_0(1+c^2)}$$

$$\Omega^2 = \frac{3g}{r_0}\,\frac{c}{1+c^2} = \frac{3g}{r_0}\cos\alpha\sin\alpha$$

or writing $P_\phi = m\rho^2\dot{\phi} = mr_0^2\omega_0$

$$\Omega^2 = \frac{3(mr_0^2\omega_0)^2}{m^2(1+c^2)r_0^4} = \frac{3\omega_0^2}{(1+c^2)} = 3\omega_0^2\sin^2\alpha$$

$$\Rightarrow \Omega = \omega_0\sin\alpha\sqrt{3}$$

The last problem illustrates one of the very few cases in which the Hamiltonian approach has any practical advantage over the Lagrangian approach for solving a simple problem. In this case, since $\mathcal{L}$ was independent of ϕ, $\mathcal{H}$ was reduced to that of a 1D problem. Instead of first writing the EOM for ρ and then eliminating $\dot{\phi}$ in favor of p_ϕ, this elimination happened at the stage of writing down $\mathcal{H}$. That makes it impossible for us to make the frequent mistake of forgetting to eliminate $\dot{\phi}$ from the ρ EOM before solving for the frequency of small oscillations w.r.t. the circular orbit $\rho = r_0$.

Another stated advantage of the Hamiltonian formalism is the ability to perform "Canonical transformations" to new variables Q and P that still obey Hamilton's equations. Let's work through Taylor's two examples of that. (Next time.)

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