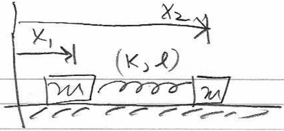
Physics 351 — Wednesday, April 18, 2018

- HW12, due either Friday 4/20 or Monday 4/23, as you prefer.
- Bill will be in DRL 3N6 today (Wed) 4–7pm.
  INSTEAD OF her usual Thursday help, Grace will be in DRL 3W2 this Sunday from 10:30am–1:30pm.
- FYI – Millie will bring her well-trained dog to class on Friday. They will sit near the front of the room.
- Some people have expressed interest in forming a study group to help review or catch up on material from this course, as the semester winds down. Learning physics really is a lot more fun when it is done cooperatively. I am happy to faciliate this — I think it may be especially beneficial to people who have fallen a bit behind and may want some camaraderie to stay on track with catching up. Even people who are caught up can benefit, I'm sure, from reviewing and talking through the ideas. Is this worth pursuing?

(This was the last thing we did on Monday.)

Morin 15.28. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let their positions be x_1 and x_2. Find $\mathcal{H}$ in terms of x_1 and x_2 and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.28



$$T = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) \qquad U = \tfrac{1}{2}k(x_1 + \ell - x_2)^2$$

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial T}{\partial \dot{x}_1} = m\dot{x}_1 \qquad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial T}{\partial \dot{x}_2} = m\dot{x}_2$$

$$\dot{x}_1 = \frac{p_1}{m} \quad \dot{x}_2 = \frac{p_2}{m} \quad \Rightarrow \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{p_1^2 + p_2^2}{2m}$$

$$\mathcal{H} = p_1\dot{x}_1 + p_2\dot{x}_2 - \mathcal{L} = \frac{p_1^2 + p_2^2}{m} - \left(\frac{p_1^2 + p_2^2}{2m} - \tfrac{1}{2}k(x_1 + \ell - x_2)^2\right)$$

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \tfrac{1}{2}k(x_1 + \ell - x_2)^2 = T + U$$

(expected result, as coords are "natural")

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{p_1}{m} \qquad \dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m}$$

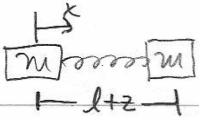
$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = -k(x_1 + \ell - x_2) \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = +k(x_1 + \ell - x_2)$$

$$\Rightarrow \ddot{x}_1 = -\frac{k}{m}(x_1 + \ell - x_2)$$

$$\ddot{x}_2 = \frac{k}{m}(x_1 + \ell - x_2)$$

Morin 15.8. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let the position of the left bead be x, and let z be the stretch of the spring (w.r.t. equilibrium). Find $\mathcal{H}$ in terms of x and z and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.8



$U = \frac{1}{2} k z^2 \qquad T = \frac{m}{2}\left(\dot{x}^2 + (\dot{x}+\dot{z})^2\right)$

$= \frac{1}{2} m \left(2\dot{x}^2 + 2\dot{x}\dot{z} + \dot{z}^2\right)$

$= m\dot{x}^2 + m\dot{x}\dot{z} + \frac{1}{2} m \dot{z}^2$

In terms of last problem's coordinates: $\left.\begin{array}{l} x_1 = x \\ x_2 = x + z + l \end{array}\right\}$ $\begin{array}{l} \checkmark \\ \neq f(t) \\ \neq f(\dot{x}) \\ \neq f(\dot{z}) \end{array}$

$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2m\dot{x} + m\dot{z}$

$P_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{x} + m\dot{z} \longrightarrow \dot{x} + \dot{z} = \frac{P_z}{m}$

$P_x - P_z = m\dot{x} \Rightarrow \dot{x} = \frac{P_x - P_z}{m}$

$m\dot{z} = P_z - m\dot{x} = P_z - (P_x - P_z) = 2P_z - P_x \Rightarrow \dot{z} = \frac{2P_z - P_x}{m}$

$$\mathcal{H} = \frac{m}{2}\left(\left(\frac{P_x - P_z}{m}\right)^2 + \left(\frac{P_z}{m}\right)^2\right) + \frac{kz^2}{2} = \frac{1}{2m}\left(P_x^2 - 2P_xP_z + 2P_z^2\right) + \frac{kz^2}{2}$$

$$\mathcal{H} = \frac{P_x^2}{2m} - \frac{P_xP_z}{m} + \frac{P_z^2}{m} + \frac{kz^2}{2}$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x} = \frac{P_x}{m} - \frac{P_z}{m} \qquad \dot{z} = \frac{\partial \mathcal{H}}{\partial P_z} = \frac{2P_z}{m} - \frac{P_x}{m}$$

$$\dot{P}_x = 0 \qquad \dot{P}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz$$

$$\left(\ddot{x} = +\frac{k}{m} z = -\frac{\ddot{z}}{2}\right) \qquad \ddot{z} = \frac{2}{m}(-kz) = -\frac{2k}{m} z$$

What do you expect the general solution to the motion to look like? (It's similar to HW problem 2 you just solved, the pendulum mounted on a frictionless horizontal rail.)

Let's try same problem one more way:

let X = cm position $= \frac{1}{2}(x_1 + x_2)$

let $z + l = x_2 - x_1$

$$\Rightarrow T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{m}{2}\right)\dot{z}^2 \quad , \quad U = \frac{1}{2}kz^2$$

$\uparrow$ M_{total} $\qquad$ $\uparrow$ μ = reduced mass

$$T = m\dot{x}^2 + \frac{1}{4}m\dot{z}^2 \qquad p_x = 2m\dot{x} \qquad p_z = \frac{1}{2}m\dot{z}$$

coords "natural" $\Rightarrow$ $\qquad \dot{x} = \frac{p_x}{2m} \qquad \dot{z} = \frac{2p_z}{m}$

$$H = T + U = m\left(\frac{p_x}{2m}\right)^2 + \frac{m}{4}\left(\frac{2p_z}{m}\right)^2 + \frac{1}{2}kz^2$$

$$H = \frac{p_x^2}{4m} + \frac{p_z^2}{m} + \frac{1}{2}kz^2$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \quad , \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{2m} \Rightarrow \ddot{x} = 0$$

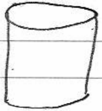
cm velocity constant

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{2p_z}{m}$$

$$\Rightarrow \ddot{z} = -\frac{2k}{m}z = -\frac{k}{\mu}z \rightarrow$$ relative coordinate oscillates sinusoidally

Taylor 13.13. Consider a particle of mass m constrained to move on a frictionless cylinder or radius R, given by the equation $\rho = R$ in (ρ, ϕ, z) coords. The mass is subject to force $\boldsymbol{F} = -kr\hat{\boldsymbol{r}}$, where k is a positive constant, r is distance from the origin, and $\hat{\boldsymbol{r}}$ points away from the origin. Using z and ϕ as generalized coordinates, find $\mathcal{H}$, write down Hamilton's equations, and describe the motion.

$$\vec{F} = -kr\hat{r} \Rightarrow U = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(\rho^2+z^2)$$
$$= \tfrac{1}{2}k(R^2+z^2)$$



so might as well write $U = \frac{1}{2}kz^2$

$$T = \tfrac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2)$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$$

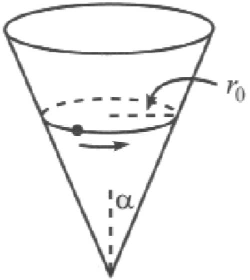
$$\mathcal{H} = T+U = \frac{p_z^2}{2m} + \tfrac{1}{2}mR^2\left(\frac{p_\phi}{mR^2}\right)^2 + \tfrac{1}{2}kz^2 = \frac{p_z^2}{2m} + \frac{p_\phi^2}{2mR^2} + \tfrac{1}{2}kz^2$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \Rightarrow \ddot{z} = -\frac{k}{m}z$$

Here's a familiar problem from HW5. Let's work through it it using Hamilton's equations instead.

3. A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \qquad U = mgz$$

$$\rho = z\tan\alpha \Rightarrow z = \rho\cot\alpha \equiv c\rho$$

Generalized coordinates : ρ, ϕ ("natural")

$$\left[z = c\rho \qquad x = \rho\cos\phi \qquad y = \rho\sin\phi\right]$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + c^2\dot{\rho}^2\right) \qquad U = mgc\rho$$

$$\mathcal{H} = T + U = \frac{m}{2}\left((1+c^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mgc\rho$$

need to rewrite in terms of P_ρ and P_ϕ

$$P_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\rho^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{P_\phi}{m\rho^2}$$

$$P_\rho = \frac{\partial T}{\partial\dot{\rho}} = m(1+c^2)\dot{\rho} \Rightarrow \dot{\rho} = \frac{P_\rho}{m(1+c^2)}$$

$$\mathcal{H} = \frac{m}{2}\left((1+c^2)\left(\frac{P_\rho}{m(1+c^2)}\right)^2 + \rho^2\left(\frac{P_\phi}{m\rho^2}\right)^2\right) + mgc\rho$$

$$\mathcal{H} = \frac{P_\rho^2}{2m(1+c^2)} + \frac{P_\phi^2}{2m\rho^2} + mgc\rho$$

We wrote down $\mathcal{H}$ and stopped there until next time.

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

$p_\phi \equiv$ const. (ϕ is ignorable/cyclic)

$\longrightarrow$ we have a 1D problem

$$\dot{p}_\rho = \frac{\partial \mathcal{H}}{\partial \rho} = -\frac{p_\phi^2}{m\rho^3} + mgc \quad \Bigg| \quad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial p_\rho} = \frac{-p_\rho}{m(1+c^2)}$$

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right)$$

$$\ddot{\rho} = 0 \Rightarrow mgc = \frac{p_\phi^2}{mr_0^3} \Rightarrow \boxed{r_0^3 = \frac{p_\phi^2}{m^2gc}} = \frac{(mr_0^2\omega_0)^2}{m^2gc} \Rightarrow \omega = \sqrt{\frac{gc}{r_0}}$$

Consider small oscillations of ρ about r_0.

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right) \equiv f(\rho)$$

$$f(r_0+\varepsilon) = f(r_0) + \varepsilon f'(r_0) + \cancel{\mathcal{O}(\varepsilon^2)}$$

$$f(r_0) = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m r_0^3} - mgc\right) = 0$$

$$f'(\rho) = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m}\right)\left(\frac{-3}{\rho^4}\right) = -\frac{3p_\phi^2}{m^2(1+c^2)\rho^4}$$

$$f'(r_0) = -\frac{3p_\phi^2}{m^2(1+c^2)r_0^4}$$

$$\rho = r_0 + \varepsilon \Rightarrow \ddot{\varepsilon} = \ddot{\rho}$$

$$\ddot{\varepsilon} = -\frac{3p_\phi^2}{m^2(1+c^2)r_0^4}\,\varepsilon \quad \Rightarrow \Omega^2 = \frac{3p_\phi^2}{m^2(1+c^2)r_0^4} = \frac{3mgc}{mr_0(1+c^2)}$$

$$\Omega^2 = \frac{3g}{r_0}\,\frac{c}{1+c^2} = \frac{3g}{r_0}\cos\alpha\sin\alpha$$

or writing $p_\phi = m\rho^2\dot{\phi} = mr_0^2\omega_0$

$$\Omega^2 = \frac{3(mr_0^2\omega_0)^2}{m^2(1+c^2)r_0^4} = \frac{3\omega_0^2}{(1+c^2)} = 3\omega_0^2\sin^2\alpha$$

$$\Rightarrow \Omega = \omega_0\sin\alpha\sqrt{3}$$

The last problem illustrates one of the very few cases in which the Hamiltonian approach has any practical advantage over the Lagrangian approach for solving a simple problem. In this case, since $\mathcal{L}$ was independent of ϕ, $\mathcal{H}$ was reduced to that of a 1D problem. Instead of first writing the EOM for ρ and then eliminating $\dot{\phi}$ in favor of p_ϕ, this elimination happened at the stage of writing down $\mathcal{H}$. That makes it impossible for us to make the frequent mistake of forgetting to eliminate $\dot{\phi}$ from the ρ EOM before solving for the frequency of small oscillations w.r.t. the circular orbit $\rho = r_0$.

Another stated advantage of the Hamiltonian formalism is the ability to perform "Canonical transformations" to new variables Q and P that still obey Hamilton's equations. Let's work through Taylor's two examples of that.

Taylor 13.24. Here is a simple example of a canonical transformation that illustrates how the Hamiltonian formalism lets one mix up the q's and the p's. Consider a system with one DOF and $\mathcal{H} = \mathcal{H}(q, p)$. The EOMs are the usual Hamilton's equations:

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

Now consider new coordinates in phase space defined as $Q = p$ and $P = -q$. Show that the EOMs for Q and P are

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

that is, the Hamiltonian formalism applies equally to the new choice of coordinates where we have exchanged the roles of position and momentum.

$$\mathcal{H} = \mathcal{H}(q, p)$$

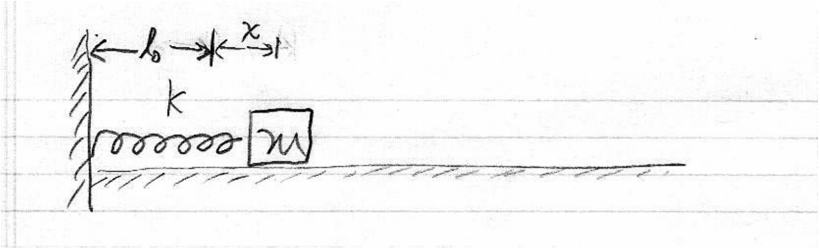
$$Q = p \qquad\qquad P = -q$$

$$\dot{Q} = \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial \mathcal{H}}{\partial(-q)} = \frac{\partial \mathcal{H}}{\partial P}$$

$$\dot{P} = -\dot{q} = -\frac{\partial \mathcal{H}}{\partial p} = -\frac{\partial \mathcal{H}}{\partial Q}$$

$$\Rightarrow \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \qquad\qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

(Intermezzo)



Write $\mathcal{H}$ for this familiar system (mass m attached to wall via spring k, where x denotes how far the spring is stretched w.r.t. its equilibrium length; no friction).

Then write Hamilton's equations of motion.

Then substitute $k = m\omega_o^2$, which shows how you'll usually see $\mathcal{H}$ written for an oscillator when you study quantum mechanics.

Taylor 13.25. Here is another example of a canonical transformation, which is still too simple to be of any real use, but illustrates the power of these changes of coordinates.

(a) Consider a system with one DOF and $\mathcal{H} = \mathcal{H}(q, p)$. Define new coordinates Q and P such that

$$q = \sqrt{2P}\,\sin(Q) \qquad\qquad p = \sqrt{2P}\,\cos(Q)$$

Prove that if $\partial\mathcal{H}/\partial q = -\dot{p}$ and $\partial\mathcal{H}/\partial p = \dot{q}$, then it automatically follows that $\partial\mathcal{H}/\partial Q = -\dot{P}$ and $\partial\mathcal{H}/\partial P = \dot{Q}$.

In other words, Hamilton's equations apply just as well to the new coordinates as to the old.

(b) Show that $\mathcal{H}$ for a 1D harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$.

(Stay tuned for parts (c) and (d)!)

$$q = \sqrt{2P}\,\sin(Q) \qquad p = \sqrt{2P}\,\cos(Q)$$

$$\frac{\partial}{\partial Q} H(q(Q,P), p(Q,P)) = \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q}$$

$$= (-\dot{p})(\sqrt{2P}\,\cos(Q)) + (\dot{q})(-\sqrt{2P}\,\sin(Q))$$

$$\dot{p} = -(2P)^{1/2}\sin(Q)\,\dot{Q} + (2P)^{-\frac{1}{2}}\cos(Q)\,\dot{P}$$

$$\dot{q} = (2P)^{1/2}\cos(Q)\,\dot{Q} + (2P)^{-1/2}\sin(Q)\,\dot{P}$$

$$\frac{\partial H}{\partial Q} = \left(\sqrt{2P}\,\sin(Q)\,\dot{Q} - \frac{1}{\sqrt{2P}}\cos(Q)\,\dot{P}\right)\sqrt{2P}\,\cos(Q)$$

$$- \left(\sqrt{2P}\,\cos(Q)\,\dot{Q} + \frac{1}{\sqrt{2P}}\sin(Q)\,\dot{P}\right)\sqrt{2P}\,\sin(Q)$$

$$= 2P\sin(Q)\cos(Q)\,\dot{Q} - \cos^2(Q)\,\dot{P} - 2P\cos(Q)\sin(Q)\,\dot{Q} - \sin^2(Q)\,\dot{P}$$

$$= \boxed{-\dot{P} = \frac{\partial H}{\partial Q}} \checkmark$$

$$\dot{p} = -(2P)^{1/2}\sin(Q)\dot{Q} + (2P)^{-\frac{1}{2}}\cos(Q)\dot{P}$$

$$\dot{q} = (2P)^{1/2}\cos(Q)\dot{Q} + (2P)^{-1/2}\sin(Q)\dot{P}$$

$$\frac{\partial}{\partial P}H = \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} = (-\dot{p})\left(\frac{\sin Q}{\sqrt{2P}}\right) + (\dot{q})\left(\frac{\cos Q}{\sqrt{2P}}\right)$$

$$= \left(\sqrt{2P}\sin(Q)\dot{Q} - \frac{\cos Q}{\sqrt{2P}}\dot{P}\right)\frac{\sin Q}{\sqrt{2P}} + \left(\sqrt{2P}\cos(Q)\dot{Q} + \frac{\sin Q}{\sqrt{2P}}\dot{P}\right)\frac{\cos Q}{\sqrt{2P}}$$

$$= \sin^2(Q)\dot{Q} - \frac{\cos Q\sin Q}{2P}\dot{P} + \cos^2(Q)\dot{Q} + \frac{\sin Q\cos Q}{2P}\dot{P} = \boxed{\dot{Q} = \frac{\partial H}{\partial P}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \rightarrow \frac{1}{2}(q^2+p^2) \quad \text{with } m=1,\ k=1$$

$$q = \sqrt{2P}\,\sin(Q) \qquad\qquad p = \sqrt{2P}\,\cos(Q)$$

(b) Show that $\mathcal{H}$ for a 1D harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$.

(c) Show that if you rewrite $\mathcal{H}$ in terms of Q and P, then Q is ignorable. What is P ?

(d) Solve the Hamiltonian equation for $Q(t)$ and verify that (when rewritten for q) the solution gives the expected behavior.

$\mathcal{H} = \frac{1}{2}(q^2+p^2) = \frac{1}{2}\left(2P\sin^2(Q) + 2P\cos^2(Q)\right) = P$

P is clearly the energy. Q is "ignorable" now.

Hamilton's equations: $\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 \rightarrow Q = t + \delta$

$\rightarrow q = \sqrt{2E}\,\sin(t+\delta)$ which agrees with

the more familiar expression

$q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin(\omega t + \delta)$ if $m = k = \omega = 1$.

$\left(E = \frac{1}{2}m\omega^2 A^2 \rightarrow \sqrt{\frac{2E}{m\omega^2}} = A\right)$

Note: Being more careful with units, you find $\mathcal{H} = \omega P$, so P actually has dimensions of **action** (same dimensions as angular momentum), not of energy. This C.T. shows the simplest example of transforming to so-called "action-angle" variables. Q is the phase, $\phi = \omega t + \delta$, a.k.a. "angle".

Recap Taylor's "Canonical transformation" example: For SHO,

$$\mathcal{H}(q,p) = T + U = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

which we know we can solve using $\dot{q} = \partial\mathcal{H}/\partial p$ and $\dot{p} = -\partial\mathcal{H}/\partial q$. By inspired guess, **transform** to variables P and Q where

$$p = \sqrt{2m\omega P}\,\cos Q \qquad\qquad q = \sqrt{\frac{2P}{m\omega}}\,\sin Q$$

This lets us rewrite $\mathcal{H}$ as

$$\mathcal{H}(Q,P) = \frac{(\sqrt{2m\omega P}\,\cos Q)^2}{2m} + \frac{1}{2}m\omega^2\left(\sqrt{\frac{2P}{m\omega}}\,\sin Q\right)^2 = \omega P$$

and Hamilton's equations give

$$\dot{P} = -\frac{\partial\mathcal{H}}{\partial Q} = 0 \qquad\qquad \dot{Q} = \frac{\partial\mathcal{H}}{\partial P} = \omega \;\rightarrow\; Q = \omega t + \delta$$

i.e. we transform the problem into one whose solution is trivial.

When you include the constants, m, k, ω, etc., the math is a bit messy, so I'll just leave it here in the notes.

$$p = \sqrt{2m\omega P}\,\cos Q \qquad q = \sqrt{\frac{2P}{m\omega}}\,\sin Q$$

$$\dot p = \sqrt{\frac{m\omega}{2P}}(\cos Q)\dot P - \sqrt{2m\omega P}\,(\sin Q)\dot Q \qquad \frac{\partial p}{\partial Q} = -\sqrt{2m\omega P}\,\sin Q$$

$$\dot q = \frac{\sin Q}{\sqrt{m\omega P}}\dot P + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot Q \qquad \frac{\partial q}{\partial Q} = \sqrt{\frac{2P}{m\omega}}\cos Q$$

$$\frac{\partial H}{\partial Q} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q} = (-\dot p)\frac{\partial q}{\partial Q} + (\dot q)\frac{\partial p}{\partial Q}$$

$$= -\sqrt{\frac{m\omega}{2P}}(\cos Q)\dot P\sqrt{\frac{2P}{m\omega}}\cos Q + \sqrt{2m\omega P}\,(\sin Q)\dot Q\sqrt{\frac{2P}{m\omega}}\cos Q$$

$$+ \frac{\sin Q}{\sqrt{m\omega P}}\dot P\left(-\sqrt{2m\omega P}\,\sin Q\right) + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot Q\left(-\sqrt{2m\omega P}\,\sin Q\right)$$

$$= -\dot P\left(\cos^2 Q + \sin^2 Q\right) = -\dot P \rightarrow \boxed{\frac{\partial H}{\partial Q} = -\dot P}$$

$$\frac{\partial H}{\partial P} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} = (-\dot p)\frac{\partial q}{\partial P} + (\dot q)\frac{\partial p}{\partial P}$$

$$= -\left[\sqrt{\frac{m\omega}{2P}}(\cos Q)\dot P - \sqrt{2m\omega P}\,(\sin Q)\dot Q\right]\frac{\sin Q}{\sqrt{2m\omega P}} \qquad \frac{\partial q}{\partial P} = \frac{\sin Q}{\sqrt{2m\omega P}}$$

$$+ \left[\frac{\sin Q}{\sqrt{m\omega P}}\dot P + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot Q\right]\sqrt{\frac{m\omega}{2P}}\cos Q \qquad \frac{\partial p}{\partial P} = \sqrt{\frac{m\omega}{2P}}\cos Q$$

$$= \dot Q\left(\sin^2 Q + \cos^2 Q\right) = \dot Q$$

Physics 351 — Wednesday, April 18, 2018

- HW12, due either Friday 4/20 or Monday 4/23, as you prefer.
- Bill will be in DRL 3N6 today (Wed) 4–7pm.
  INSTEAD OF her usual Thursday help, Grace will be in DRL 3W2 this Sunday from 10:30am–1:30pm.
- FYI – Millie will bring her well-trained dog to class on Friday. They will sit near the front of the room.