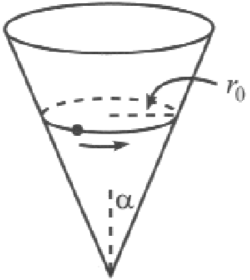
Physics 351 — Friday, April 20, 2018

- Turn in HW12 either today or Monday, as you prefer.
- **INSTEAD OF** her usual Thursday help, Grace will be in DRL 3W2 this Sunday from 10:30am–1:30pm.
- FYI – Millie will bring her well-trained dog to class today. They will sit near the front of the room.
- Some people have expressed interest in forming a study group to help review or catch up on material from this course, as the semester winds down. Learning physics really is a lot more fun when it is done cooperatively. To try to faciliate this, I created a Canvas discussion area, but so far nobody has followed up.
- For Monday, there is a required reading by Feynman on the deep connection between Lagrangian classical mechanics and Feynman's path-integral formulation of quantum mechanics.
- Required reading for next Wednesday: the two chapters on fluids from the Feynman Lectures on Physics. Reading the Feynman Lectures is a fun way to deepen your understanding of introductory physics. They are a work of art.

Here's a familiar problem from HW5 (and in varied form from HW12). Let's work through it it using Hamilton's equations.

3. A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \qquad U = mgz$$

$$\rho = z\tan\alpha \Rightarrow z = \rho\cot\alpha \equiv c\rho$$

Generalized coordinates: ρ, ϕ ("natural")

$$\left[z = c\rho \qquad x = \rho\cos\phi \qquad y = \rho\sin\phi\right]$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + c^2\dot{\rho}^2\right) \qquad U = mgc\rho$$

$$\mathcal{H} = T + U = \frac{m}{2}\left((1+c^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mgc\rho$$

need to rewrite in terms of p_ρ and p_ϕ

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\rho^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{m\rho^2}$$

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = m(1+c^2)\dot{\rho} \Rightarrow \dot{\rho} = \frac{p_\rho}{m(1+c^2)}$$

$$\mathcal{H} = \frac{m}{2}\left((1+c^2)\left(\frac{p_\rho}{m(1+c^2)}\right)^2 + \rho^2\left(\frac{p_\phi}{m\rho^2}\right)^2\right) + mgc\rho$$

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

We wrote down $\mathcal{H}$ and stopped there until next time.

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

$p_\phi \equiv$ const. (ϕ is ignorable/cyclic)

$\longrightarrow$ we have a 1D problem

$$\dot{p}_\rho = \frac{\partial \mathcal{H}}{\partial \rho} = -\frac{p_\phi^2}{m\rho^3} + mgc \quad \Bigg| \quad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial p_\rho} = \frac{-p_\rho}{m(1+c^2)}$$

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right)$$

$$\ddot{\rho} = 0 \Rightarrow mgc = \frac{p_\phi^2}{mr_0^3} \Rightarrow \boxed{r_0^3 = \frac{p_\phi^2}{m^2gc}} = \frac{(mr_0^2\omega_0)^2}{m^2gc} \Rightarrow \omega = \sqrt{\frac{gc}{r_0}}$$

Consider small oscillations of ρ about r_0.

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right) \equiv f(\rho)$$

$$f(r_0+\varepsilon) = f(r_0) + \varepsilon f'(r_0) + \cancel{\mathcal{O}(\varepsilon^2)}$$

$$f(r_0) = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{mr_0^3} - mgc\right) = 0$$

$$f'(\rho) = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m}\right)\left(\frac{-3}{\rho^4}\right) = -\frac{3p_\phi^2}{m^2(1+c^2)\rho^4}$$

$$f'(r_0) = -\frac{3p_\phi^2}{m^2(1+c^2)r_0^4}$$

$$\rho = r_0 + \varepsilon \Rightarrow \ddot{\varepsilon} = \ddot{\rho}$$

$$\ddot{\varepsilon} = -\frac{3p_\phi^2}{m^2(1+c^2)r_0^4}\,\varepsilon \quad \Rightarrow \Omega^2 = \frac{3p_\phi^2}{m^2(1+c^2)r_0^4} = \frac{3mgc}{mr_0(1+c^2)}$$

$$\Omega^2 = \frac{3g}{r_0}\,\frac{c}{1+c^2} = \frac{3g}{r_0}\cos\alpha\sin\alpha$$

or writing $p_\phi = m\rho^2\dot{\phi} = mr_0^2\omega_0$

$$\Omega^2 = \frac{3(mr_0^2\omega_0)^2}{m^2(1+c^2)r_0^4} = \frac{3\omega_0^2}{(1+c^2)} = 3\omega_0^2\sin^2\alpha$$

$$\Rightarrow \Omega = \omega_0\sin\alpha\sqrt{3}$$

The last problem illustrates one of the very few cases in which the Hamiltonian approach has any practical advantage over the Lagrangian approach for solving a simple problem. In this case, since $\mathcal{L}$ was independent of ϕ, $\mathcal{H}$ was reduced to that of a 1D problem. Instead of first writing the EOM for ρ and then eliminating $\dot{\phi}$ in favor of p_ϕ, this elimination happened at the stage of writing down $\mathcal{H}$. That makes it impossible for us to make the frequent mistake of forgetting to eliminate $\dot{\phi}$ from the ρ EOM before solving for the frequency of small oscillations w.r.t. the circular orbit $\rho = r_0$.

Another stated advantage of the Hamiltonian formalism is the ability to perform "Canonical transformations" to new variables Q and P that still obey Hamilton's equations. Let's work through Taylor's two examples of that.

Taylor 13.24. Here is a simple example of a canonical transformation that illustrates how the Hamiltonian formalism lets one mix up the q's and the p's. Consider a system with one DOF and $\mathcal{H} = \mathcal{H}(q, p)$. The EOMs are the usual Hamilton's equations:

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

Now consider new coordinates in phase space defined as $Q = p$ and $P = -q$. Show that the EOMs for Q and P are

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

that is, the Hamiltonian formalism applies equally to the new choice of coordinates where we have exchanged the roles of position and momentum.

$$\mathcal{H} = \mathcal{H}(q, p)$$

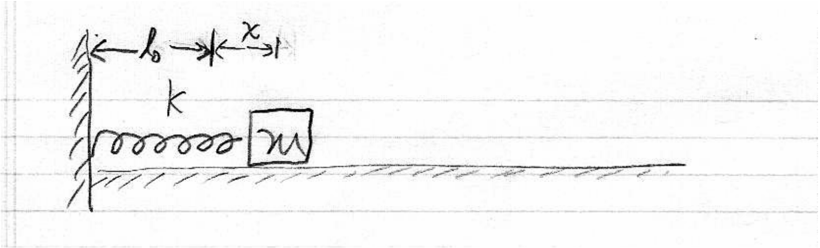
$$Q = p \qquad\qquad P = -q$$

$$\dot{Q} = \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial \mathcal{H}}{\partial(-q)} = \frac{\partial \mathcal{H}}{\partial P}$$

$$\dot{P} = -\dot{q} = -\frac{\partial \mathcal{H}}{\partial p} = -\frac{\partial \mathcal{H}}{\partial Q}$$

$$\Rightarrow \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \qquad\qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

(Intermezzo)



Write $\mathcal{H}$ for this familiar system (mass m attached to wall via spring k, where x denotes how far the spring is stretched w.r.t. its equilibrium length; no friction).

Then write Hamilton's equations of motion.

Then substitute $k = m\omega_o^2$, which shows how you'll usually see $\mathcal{H}$ written for an oscillator when you study quantum mechanics.

Taylor 13.25. Here is another example of a canonical transformation, which is still too simple to be of any real use, but illustrates the power of these changes of coordinates.

(a) Consider a system with one DOF and $\mathcal{H} = \mathcal{H}(q, p)$. Define new coordinates Q and P such that

$$q = \sqrt{2P}\,\sin(Q) \qquad\qquad p = \sqrt{2P}\,\cos(Q)$$

Prove that if $\partial\mathcal{H}/\partial q = -\dot{p}$ and $\partial\mathcal{H}/\partial p = \dot{q}$, then it automatically follows that $\partial\mathcal{H}/\partial Q = -\dot{P}$ and $\partial\mathcal{H}/\partial P = \dot{Q}$.

In other words, Hamilton's equations apply just as well to the new coordinates as to the old.

(b) Show that $\mathcal{H}$ for a 1D harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$.

(Stay tuned for parts (c) and (d)!)

$$q = \sqrt{2P}\,\sin(Q) \qquad p = \sqrt{2P}\,\cos(Q)$$

$$\frac{\partial}{\partial Q} H(q(Q,P), p(Q,P)) = \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q}$$

$$= (-\dot{p})(\sqrt{2P}\,\cos(Q)) + (\dot{q})(-\sqrt{2P}\,\sin(Q))$$

$$\dot{p} = -(2P)^{1/2}\sin(Q)\,\dot{Q} + (2P)^{-\frac{1}{2}}\cos(Q)\,\dot{P}$$

$$\dot{q} = (2P)^{1/2}\cos(Q)\,\dot{Q} + (2P)^{-1/2}\sin(Q)\,\dot{P}$$

$$\frac{\partial H}{\partial Q} = \left(\sqrt{2P}\,\sin(Q)\,\dot{Q} - \frac{1}{\sqrt{2P}}\cos(Q)\,\dot{P}\right)\sqrt{2P}\,\cos(Q)$$

$$- \left(\sqrt{2P}\,\cos(Q)\,\dot{Q} + \frac{1}{\sqrt{2P}}\sin(Q)\,\dot{P}\right)\sqrt{2P}\,\sin(Q)$$

$$= 2P\sin(Q)\cos(Q)\,\dot{Q} - \cos^2(Q)\,\dot{P} - 2P\cos(Q)\sin(Q)\,\dot{Q} - \sin^2(Q)\,\dot{P}$$

$$= \boxed{-\dot{P} = \frac{\partial H}{\partial Q}} \checkmark$$

$$\dot{p} = -(2P)^{1/2}\sin(Q)\dot{Q} + (2P)^{-\frac{1}{2}}\cos(Q)\dot{P}$$

$$\dot{q} = (2P)^{1/2}\cos(Q)\dot{Q} + (2P)^{-1/2}\sin(Q)\dot{P}$$

$$\frac{\partial}{\partial P}H = \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} = (-\dot{p})\left(\frac{\sin Q}{\sqrt{2P}}\right) + (\dot{q})\left(\frac{\cos Q}{\sqrt{2P}}\right)$$

$$= \left(\sqrt{2P}\sin(Q)\dot{Q} - \frac{\cos Q}{\sqrt{2P}}\dot{P}\right)\frac{\sin Q}{\sqrt{2P}} + \left(\sqrt{2P}\cos(Q)\dot{Q} + \frac{\sin Q}{\sqrt{2P}}\dot{P}\right)\frac{\cos Q}{\sqrt{2P}}$$

$$= \sin^2(Q)\dot{Q} - \frac{\cos Q\sin Q}{2P}\dot{P} + \cos^2(Q)\dot{Q} + \frac{\sin Q\cos Q}{2P}\dot{P} = \boxed{\dot{Q} = \frac{\partial H}{\partial P}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \rightarrow \frac{1}{2}(q^2+p^2) \quad \text{with } m=1,\ k=1$$

$$q = \sqrt{2P}\,\sin(Q) \qquad\qquad p = \sqrt{2P}\,\cos(Q)$$

(b) Show that $\mathcal{H}$ for a 1D harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$.

(c) Show that if you rewrite $\mathcal{H}$ in terms of Q and P, then Q is ignorable. What is P ?

(d) Solve the Hamiltonian equation for $Q(t)$ and verify that (when rewritten for q) the solution gives the expected behavior.

$$\mathcal{H} = \frac{1}{2}(q^2+p^2) = \frac{1}{2}\left(2P\sin^2(Q) + 2P\cos^2(Q)\right) = P$$

P is clearly the energy. Q is "ignorable" now.

Hamilton's equations: $\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 \rightarrow Q = t+\delta$

$\rightarrow q = \sqrt{2E}\,\sin(t+\delta)$ which agrees with

the more familiar expression

$$q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin(\omega t+\delta) \quad \text{if } m=k=\omega=1.$$

$$p(t) = \sqrt{2mE}\,\cos(\omega t+\delta)$$

$$\left(E = \tfrac{1}{2}m\omega^2A^2 \rightarrow \sqrt{\frac{2E}{m\omega^2}} = A\right)$$

Note: Being more careful with units, you find $\mathcal{H} = \omega P$, so P actually has dimensions of **action** (same dimensions as angular momentum), not of energy. This C.T. shows the simplest example of transforming to so-called "action-angle" variables. Q is the phase, $\phi = \omega t + \delta$, a.k.a. "angle".

Recap Taylor's "Canonical transformation" example: For SHO,

$$\mathcal{H}(q,p) = T + U = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

which we know we can solve using $\dot{q} = \partial\mathcal{H}/\partial p$ and $\dot{p} = -\partial\mathcal{H}/\partial q$.
By inspired guess, **transform** to variables P and Q where

$$p = \sqrt{2m\omega P}\,\cos Q \qquad\qquad q = \sqrt{\frac{2P}{m\omega}}\,\sin Q$$

This lets us rewrite $\mathcal{H}$ as

$$\mathcal{H}(Q,P) = \frac{(\sqrt{2m\omega P}\,\cos Q)^2}{2m} + \frac{1}{2}m\omega^2\left(\sqrt{\frac{2P}{m\omega}}\,\sin Q\right)^2 = \omega P$$

and Hamilton's equations give

$$\dot{P} = -\frac{\partial\mathcal{H}}{\partial Q} = 0 \qquad\qquad \dot{Q} = \frac{\partial\mathcal{H}}{\partial P} = \omega \;\rightarrow\; Q = \omega t + \delta$$

i.e. we transform the problem into one whose solution is trivial.

When you include the constants, m, k, ω, etc., the math is a bit messy, so I'll just leave it here in the notes.

$$p = \sqrt{2m\omega P}\,\cos Q \qquad q = \sqrt{\frac{2P}{m\omega}}\,\sin Q$$

$$\dot{p} = \sqrt{\frac{m\omega}{2P}}(\cos Q)\dot{P} - \sqrt{2m\omega P}(\sin Q)\dot{Q} \qquad \frac{\partial p}{\partial Q} = -\sqrt{2m\omega P}\,\sin Q$$

$$\dot{q} = \frac{\sin Q}{\sqrt{m\omega P}}\dot{P} + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot{Q} \qquad \frac{\partial q}{\partial Q} = \sqrt{\frac{2P}{m\omega}}\,\cos Q$$

$$\begin{aligned}
\frac{\partial H}{\partial Q} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q} = (-\dot{p})\frac{\partial q}{\partial Q} + (\dot{q})\frac{\partial p}{\partial Q} \\
&= -\sqrt{\frac{m\omega}{2P}}(\cos Q)\dot{P}\sqrt{\frac{2P}{m\omega}}\cos Q + \sqrt{2m\omega P}(\sin Q)\dot{Q}\sqrt{\frac{2P}{m\omega}}\cos Q \\
&\quad + \frac{\sin Q}{\sqrt{m\omega P}}\dot{P}\left(-\sqrt{2m\omega P}\sin Q\right) + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot{Q}\left(-\sqrt{2m\omega P}\sin Q\right) \\
&= -\dot{P}(\cos^2 Q + \sin^2 Q) = -\dot{P} \quad \Rightarrow \quad \boxed{\frac{\partial H}{\partial Q} = -\dot{P}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial P} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} = (-\dot{p})\frac{\partial q}{\partial P} + (\dot{q})\frac{\partial p}{\partial P} \\
&= -\left[\sqrt{\frac{m\omega}{2P}}(\cos Q)\dot{P} - \sqrt{2m\omega P}(\sin Q)\dot{Q}\right]\frac{\sin Q}{\sqrt{2m\omega P}} \qquad \frac{\partial q}{\partial P} = \frac{\sin Q}{\sqrt{2m\omega P}} \\
&\quad + \left[\frac{\sin Q}{\sqrt{m\omega P}}\dot{P} + \sqrt{\frac{2P}{m\omega}}(\cos Q)\dot{Q}\right]\sqrt{\frac{m\omega}{2P}}\cos Q \qquad \frac{\partial p}{\partial P} = \sqrt{\frac{m\omega}{2P}}\cos Q \\
&= \dot{Q}(\sin^2 Q + \cos^2 Q) = \dot{Q}
\end{aligned}$$

Legendre transform: given $F(x)$, find $G(s)$ such that $G'(s)$ is the inverse function of $F'(x)$.

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

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$F(x) = x^2 \rightarrow s = F' = 2x$
$\rightarrow x = \frac{s}{2}$

$G = sx - F = (s)\left(\frac{s}{2}\right) - \left(\frac{s}{2}\right)^2$
$= s^2/2 - s^2/4 = \boxed{s^2/4}$

check: $\frac{dG}{ds} = \frac{d}{ds}\left(\frac{s^2}{4}\right) = \frac{s}{2} = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

$F(x) = \ln x \rightarrow s = F' = \frac{1}{x}$ $\qquad G = sx - F = s\left(\frac{1}{s}\right) - \ln\left(\frac{1}{s}\right)$

$\rightarrow x = 1/s$ $\qquad = \boxed{1 + \ln(s)}$

check: $\frac{dG}{ds} = \frac{d}{ds}\left(1 + \ln s\right) = \frac{1}{s} = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

$F(x) = e^x \rightarrow s = F' = e^x$ $\quad G = sx - F = s\ln s - e^{\ln s}$

$\rightarrow x = \ln s$ $\quad = \boxed{s\ln s - s}$

check: $\frac{dG}{ds} = \frac{d}{ds}(s\ln s - s) = \ln s + s \cdot \frac{1}{s} - 1 = \ln s = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

$F(x) = x^n \rightarrow s = F' = n x^{n-1}$

$\rightarrow x = (s/n)^{1/(n-1)}$

$G = sx - F = s\left(\frac{s}{n}\right)^{\frac{1}{(n-1)}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= n\left(\frac{s}{n}\right)\left(\frac{s}{n}\right)^{\frac{1}{n-1}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= n\left(\frac{s}{n}\right)^{\frac{(n-1)+1}{n-1}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= \boxed{(n-1)\left(\frac{s}{n}\right)^{\frac{n}{n-1}} = G(s)}$

check: $\frac{dG}{ds} =$

$(n-1)\left(\frac{n}{n-1}\right)\left(\frac{s}{n}\right)^{\frac{n-(n-1)}{n-1}}\left(\frac{1}{n}\right)$

$= \left(\frac{s}{n}\right)^{\frac{1}{n-1}} = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ | $F(x) = \ln x$ | $F(x) = e^x$ | $F = x^n$

Now apply same recipe to $\mathcal{L}$:

Now take $\mathcal{L}(q, \dot{q})$ to be $F(\dot{q})$

define $p(\dot{q}) = F'(\dot{q}) = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}$

So "x" is $\dot{q}$
"s" is p
"F" is $\mathcal{L}$
"G" is $\mathcal{H}$

Since this is just a Legendre transform, $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$

Working it out in detail,

Now take $\mathcal{L}(q,\dot{q})$ to be $F(\dot{q})$

define $p(\dot{q}) = F'(\dot{q}) = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}$

So "x" is $\dot{q}$

"s" is p

"F" is $\mathcal{L}$

"G" is $\mathcal{H}$

Since this is just a Legendre transform, $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$

Working it out in detail,

$$\frac{\partial}{\partial p}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}$$

$$= \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q} \rightarrow \boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

While were at it, let's work out $\frac{\partial \mathcal{H}}{\partial q}$

Since this is just a Legendre transform, $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$

Working it out in detail,

$$\frac{\partial}{\partial p}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}$$

$$= \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q} \rightarrow \boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

While were at it, let's work out $\frac{\partial \mathcal{H}}{\partial q}$

$$\frac{\partial}{\partial q}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

$$= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - p\frac{\partial \dot{q}}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = -\dot{p}$$

$$\rightarrow \boxed{\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}}$$

Analogous example from thermal physics:

Internal energy of system (let's say number N of particles is fixed, not variable)

$$\mathcal{U} = \mathcal{U}(S, V) \Rightarrow d\mathcal{U} = T\,dS - P\,dV$$

notice that $\left(\frac{\partial \mathcal{U}}{\partial S}\right)_{\substack{\text{constant}\\ V}} = T$ $\left(\frac{\partial \mathcal{U}}{\partial V}\right)_{\substack{\text{constant}\\ S}} = -P$

For a system in contact with a heat bath, constant T, it is convenient to eliminate S in favor of T.

In analogy with

$$\mathcal{H}(p, q) = \left(\frac{\partial \mathcal{L}(\dot{q}, q)}{\partial \dot{q}}\right)\dot{q} - \mathcal{L}(\dot{q}, q)$$

we do Legendre transform

$$-F(T, V) = \left(\frac{\partial \mathcal{U}(S, V)}{\partial S}\right) S - \mathcal{U}(S, V)$$

In analogy with

$$\mathcal{H}(p,q) = \left(\frac{\partial \mathcal{L}(\dot{q},q)}{\partial \dot{q}}\right)\dot{q} - \mathcal{L}(\dot{q},q)$$

we do Legendre transform

$$-F(T,V) = \left(\frac{\partial \mathcal{U}(S,V)}{\partial S}\right)S - \mathcal{U}(S,V)$$

$$-F = (T)S - \mathcal{U} \quad \Rightarrow \quad \boxed{F = \mathcal{U} - TS}$$

then $dF = -S\,dT - P\,dV$, so $F = F(T,V)$

For a system in equilibrium with a heat bath (aka thermal reservoir), maximizing the entropy of (system + heat bath) is equivalent to minimizing the free energy of the system.

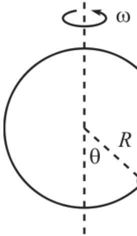
F = "Helmholtz free energy"

is Legendre transform of internal energy $\mathcal{U}$.

Question:

- When is $\mathcal{H}$ conserved (i.e. a constant of the motion)?
- When does $\mathcal{H}$ equal the total energy?
- Notice that these are two different questions.

Morin 15.11. A bead is free to slide along a frictionless hoop of radius R. The hoop is forced to rotate with constant angular speed ω around a vertical diameter. Find $\mathcal{H}$ in terms of θ and p_θ, then write down Hamilton's equations. Is $\mathcal{H}$ the energy? Is $\mathcal{H}$ conserved?



$\phi = \omega t \rightarrow \dot\phi = \omega$ $\qquad$ $(x,y) = R(\cos\omega t, \sin\omega t)$ which depends on time explicitly

$T = \frac{m}{2}\left(R^2\dot\theta^2 + (R\sin\theta)^2\omega^2\right)$ $\qquad$ $U = -mgR\cos\theta$

$\mathcal{L} = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$

$p = \frac{\partial\mathcal{L}}{\partial\dot\theta} = mR^2\dot\theta \Rightarrow \dot\theta = \frac{p}{mR^2}$

$T = \frac{1}{2}mR^2\left(\frac{p}{mR^2}\right)^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta = \frac{p^2}{2mR^2} + \frac{1}{2}mR^2\omega^2\sin^2\theta$

$\mathcal{H} = p\dot\theta - \mathcal{L} = \frac{p^2}{mR^2} - \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$

$\mathcal{H} = \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$

$\dot\theta = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{mR^2}$ $\qquad$ $\dot p = -\frac{\partial\mathcal{H}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{mR^2\omega^2}{2}\sin^2\theta + mgR\cos\theta\right)$

$\dot p = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$

$\rightarrow \ddot\theta = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$

What is the energy? $\frac{p^2}{2mR^2} + \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta \neq \mathcal{H}$

It seems that $\mathcal{H} = \text{energy} - mR^2\omega^2\sin^2\theta$

$\frac{\partial\mathcal{L}}{\partial t} = -\frac{\partial\mathcal{H}}{\partial t} = 0 \Rightarrow \mathcal{H}$ is conserved!
(but it's not the energy)

$\rightarrow$ energy $=$ const. $+ mR^2\omega^2\sin^2\theta$

(We can show by explicit calculation that $\frac{dH}{dt}=0$)

$$mR^2\ddot{\theta}\,\boxed{\dot{\theta}} = mR^2\omega^2\sin\theta\cos\theta\,\boxed{\dot{\theta}} - mgR\sin\theta\,\boxed{\dot{\theta}}$$

$$\left\{\begin{array}{l} d\theta = \dot{\theta}\,dt \\ d\dot{\theta} = \ddot{\theta}\,dt \\ \frac{1}{2}d(\dot{\theta}^2) = \dot{\theta}\ddot{\theta}\,dt \end{array}\right.$$

$$\frac{d}{dt}\left(\frac{1}{2}mR^2\dot{\theta}^2\right) = mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta} - mgR\sin\theta\,\dot{\theta}$$

$$= \frac{d}{dt}\left(\frac{p^2}{2mR^2}\right)$$

$$\rightarrow \frac{dH}{dt} = \frac{d}{dt}\left(\frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta\right)$$

$$= \frac{d}{dt}\left(\frac{p^2}{2mR^2}\right) - mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta} + mgR\sin\theta\,\dot{\theta} = 0 \checkmark$$

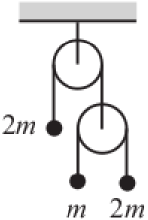
The quantity $mR^2\omega^2\sin^2\theta$ turns out to be the work done on the bead by the force of constraint that enforces $\phi = \omega t$.

$$\tau_z = \frac{dL_z}{dt} = \frac{d}{dt}\left(m(R\sin\theta)^2\omega\right) = 2mR^2\omega\sin\theta\cos\theta\,\dot{\theta}$$

Power delivered by this torque $= \frac{d(\text{work})}{dt} = \omega\tau_z = 2mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta}$

$$= \frac{d}{dt}\left(mR^2\omega^2\sin^2\theta\right)$$

Morin 15.7. Consider the Atwood machine shown in the figure. Let x and y be the vertical positions of the middle mass and right mass, respectively, with upward taken to be positive. Find $\mathcal{H}$ in terms of x and y and their conjugate momenta, then write down the four Hamilton's equations.



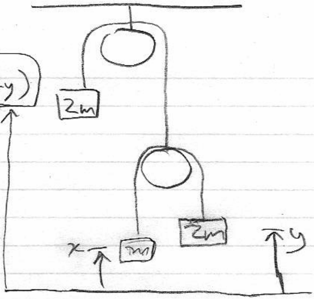
(The solution to this problem turns out to be not especially illuminating. But it does illustrate how tedious the Hamiltonian method can be for solving problems that are straightforward using the Lagrangian method.)

Morin writes: "If you want to demonstrate how the Hamiltonian method can be monumentally more cumbersome than the Lagrangian method, you can try to solve this problem in the case of three general masses, m_1, m_2, m_3." ♣

$\boxed{\text{const.} - \tfrac{1}{2}(x+y)}$

$$U = mgx + 2mgy - 2mg\,\frac{x+y}{2}$$

$$= mgx + 2mgy - mg(x+y)$$



$$U = mgy$$

$$T = \frac{m}{2}\dot{x}^2 + m\dot{y}^2 + \left(\frac{2m}{2}\right)\left(\frac{\dot{x}+\dot{y}}{2}\right)^2$$

$$= \frac{m}{2}\dot{x}^2 + m\dot{y}^2 + \frac{m}{4}\left(\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2\right) = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2$$

$$\mathcal{L} = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2 - mgy$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{3}{2}m\dot{x} + \frac{1}{2}m\dot{y}$$

$$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{1}{2}m\dot{x} + \frac{5}{2}m\dot{y}$$

$$p_x - 3p_y = \left(\frac{1}{2} - \frac{15}{2}\right)m\dot{y} = -7m\dot{y} \Rightarrow \dot{y} = \frac{3p_y - p_x}{7m}$$

$$-5p_x + p_y = \left(-\frac{15}{2} + \frac{1}{2}\right)m\dot{x} = -7m\dot{x} \Rightarrow \dot{x} = \frac{5p_x - p_y}{7m}$$

$$\mathcal{L} = \tfrac{3}{4} m\dot{x}^2 + \tfrac{1}{2} m\dot{x}\dot{y} + \tfrac{5}{4} m\dot{y}^2 - mgy$$

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \tfrac{3}{2} m\dot{x} + \tfrac{1}{2} m\dot{y}$$

$$P_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \tfrac{1}{2} m\dot{x} + \tfrac{5}{2} m\dot{y}$$

$$P_x - 3P_y = \left(\tfrac{1}{2} - \tfrac{15}{2}\right) m\dot{y} = -7m\dot{y} \Rightarrow \dot{y} = \frac{3P_y - P_x}{7m}$$

$$-5P_x + P_y = \left(-\tfrac{15}{2} + \tfrac{1}{2}\right) m\dot{x} = -7m\dot{x} \Rightarrow \dot{x} = \frac{5P_x - P_y}{7m}$$

$$T = \frac{3m}{4}\left(\frac{5P_x - P_y}{7m}\right)^2 + \frac{m}{2}\left(\frac{5P_x - P_y}{7m}\right)\left(\frac{3P_y - P_x}{7m}\right) + \frac{5m}{4}\left(\frac{3P_y - P_x}{7m}\right)^2$$

$$T = \frac{1}{14m}\left(5P_x^2 - 2P_xP_y + 3P_y^2\right) \qquad U = mgy$$

$$H = T + U = \left(\frac{1}{14m}\right)\left(5P_x^2 - 2P_xP_y + 3P_y^2\right) + mgy$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{14m}\left(10P_x - 2P_y\right) = \frac{5P_x - P_y}{7m}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{14m}\left(-2P_x + 6P_y\right) = \frac{3P_y - P_x}{7m}$$

we knew that already

$$\dot{P}_x = -\frac{\partial H}{\partial x} = 0 \qquad \ddot{x} = -\frac{\dot{P}_y}{7m} = \boxed{\frac{g}{7} = \ddot{x}}$$

$$\dot{P}_y = -\frac{\partial H}{\partial y} = -mg \qquad \ddot{y} = \frac{3\dot{P}_y}{7m} = \boxed{-\frac{3}{7}g = \ddot{y}}$$

$$\frac{\ddot{x} + \ddot{y}}{2} = -\frac{g}{7}$$

Physics 351 — Friday, April 20, 2018

- Turn in HW12 either today or Monday, as you prefer.
- **INSTEAD OF** her usual Thursday help, Grace will be in DRL 3W2 this Sunday from 10:30am–1:30pm.
- FYI – Millie will bring her well-trained dog to class today. They will sit near the front of the room.
- Some people have expressed interest in forming a study group to help review or catch up on material from this course, as the semester winds down. Learning physics really is a lot more fun when it is done cooperatively. To try to faciliate this, I created a Canvas discussion area, but so far nobody has followed up.
- For Monday, there is a required reading by Feynman on the deep connection between Lagrangian classical mechanics and Feynman's path-integral formulation of quantum mechanics.
- Required reading for next Wednesday: the two chapters on fluids from the Feynman Lectures on Physics. Reading the Feynman Lectures is a fun way to deepen your understanding of introductory physics. They are a work of art.