Physics 351 — Monday, April 23, 2018

- Turn in HW12. Last one! Hooray!
- Last day to turn in XC is Sunday, May 6 (three days after the exam). For the few people who did Perusall (sorry!), I will factor that in as XC.
- There is interest in forming study groups to help review or catch up on material from this course, as the semester winds down. Learning physics really is a lot more fun when it is done cooperatively. To try to faciliate this, I created a Canvas discussion area, but so far **only one person** has followed up.
- Don't forget the last two readings: Feynman/Hibbs (on Canvas) for today; and Feynman's two lectures on fluids, for Wednesday.
- I meant to finish Friday by saying that whereas Taylor's first example of a canonical transformation was, in effect, a 90° rotation in (p,q) space, his second example was effectively a Cartesian-to-polar transformation of (p,q) space.

Legendre transform: given $F(x)$, find $G(s)$ such that $G'(s)$ is the inverse function of $F'(x)$.

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx} x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

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$F(x) = x^2 \rightarrow s = F' = 2x$
$\rightarrow x = \frac{s}{2}$

$G = sx - F = (s)\left(\frac{s}{2}\right) - \left(\frac{s}{2}\right)^2$
$= s^2/2 - s^2/4 = \boxed{s^2/4}$

check: $\frac{dG}{ds} = \frac{d}{ds}\left(\frac{s^2}{4}\right) = \frac{s}{2} = x$ ✓

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Try 4 examples that Morin worked out for us:

$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

$F(x) = \ln x \rightarrow s = F' = \frac{1}{x}$ $\quad G = sx - F = s\left(\frac{1}{s}\right) - \ln\left(\frac{1}{s}\right)$

$\rightarrow x = 1/s$ $\quad = \boxed{1 + \ln(s)}$

check: $\frac{dG}{ds} = \frac{d}{ds}\left(1 + \ln s\right) = \frac{1}{s} = x$ ✓

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$F(x) = e^x \rightarrow s = F' = e^x$
$\rightarrow x = \ln s$

$G = sx - F = s\ln s - e^{\ln s}$
$= \boxed{s\ln s - s}$

check: $\frac{dG}{ds} = \frac{d}{ds}(s\ln s - s) = \ln s + s \cdot \frac{1}{s} - 1 = \ln s = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

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$F(x) = x^2$ } $F(x) = \ln x$ } $F(x) = e^x$ } $F = x^n$

$F(x) = x^n \rightarrow s = F' = nx^{n-1}$

$\rightarrow x = (s/n)^{1/(n-1)}$

$G = sx - F = s\left(\frac{s}{n}\right)^{\frac{1}{(n-1)}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= n\left(\frac{s}{n}\right)\left(\frac{s}{n}\right)^{\frac{1}{n-1}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= n\left(\frac{s}{n}\right)^{\frac{(n-1)+1}{n-1}} - \left(\frac{s}{n}\right)^{\frac{n}{n-1}}$

$= \boxed{(n-1)\left(\frac{s}{n}\right)^{\frac{n}{n-1}} = G(s)}$

check: $\frac{dG}{ds} =$

$(n-1)\left(\frac{n}{n-1}\right)\left(\frac{s}{n}\right)^{\frac{n-(n-1)}{n-1}}\left(\frac{1}{n}\right)$

$= \left(\frac{s}{n}\right)^{\frac{1}{n-1}} = x$ ✓

Legendre transform: define $s = F'(x) = \frac{dF(x)}{dx}$ "slope"

seek $G(s)$ such that $\frac{dG(s)}{ds} = x(s)$

recipe: $G = sx - F = F'x - F = \frac{dF}{dx}x - F$

Try 4 examples that Morin worked out for us:

$F(x) = x^2 \;\}\; F(x) = \ln x \;\}\; F(x) = e^x \;\}\; F = x^n$

Now apply same recipe to $\mathcal{L}$:

Now take $\mathcal{L}(q, \dot{q})$ to be $F(\dot{q})$

define $p(\dot{q}) = F'(\dot{q}) = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}$

So "x" is $\dot{q}$
"s" is p
"F" is $\mathcal{L}$
"G" is $\mathcal{H}$

Since this is just a Legendre transform, $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$

Working it out in detail,

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Working it out in detail,

$$\frac{\partial}{\partial p}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}$$

$$= \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q} \rightarrow \boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

While we're at it, let's work out $\frac{\partial \mathcal{H}}{\partial q}$

Since this is just a Legendre transform, $\frac{\partial H}{\partial p} = \dot{q}$

Working it out in detail,

$$\frac{\partial}{\partial p}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}$$

$$= \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q} \rightarrow \boxed{\frac{\partial H}{\partial p} = \dot{q}}$$

While we're at it, let's work out $\frac{\partial H}{\partial q}$

$$\frac{\partial}{\partial q}\left(p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))\right) = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

$$= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - p\frac{\partial \dot{q}}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = -\dot{p}$$

$$\rightarrow \boxed{\frac{\partial H}{\partial q} = -\dot{p}}$$

Analogous example from thermal physics:

Internal energy of system (let's say number N of particles is fixed, not variable)

$$\mathcal{U} = \mathcal{U}(S, V) \Rightarrow d\mathcal{U} = T\,dS - P\,dV$$

notice that $\left(\frac{\partial \mathcal{U}}{\partial S}\right)_{\substack{\text{constant}\\ V}} = T$; $\left(\frac{\partial \mathcal{U}}{\partial V}\right)_{\substack{\text{constant}\\ S}} = -P$

For a system in contact with a heat bath, constant T, it is convenient to eliminate S in favor of T.

In analogy with

$$\mathcal{H}(p, q) = \left(\frac{\partial \mathcal{L}(\dot{q}, q)}{\partial \dot{q}}\right)\dot{q} - \mathcal{L}(\dot{q}, q)$$

we do Legendre transform

$$-F(T, V) = \left(\frac{\partial \mathcal{U}(S, V)}{\partial S}\right) S - \mathcal{U}(S, V)$$

In analogy with

$$\mathcal{H}(p,q) = \left(\frac{\partial \mathcal{L}(\dot{q},q)}{\partial \dot{q}}\right)\dot{q} - \mathcal{L}(\dot{q},q)$$

we do Legendre transform

$$-F(T,V) = \left(\frac{\partial U(S,V)}{\partial S}\right)S - U(S,V)$$

$$-F = (T)S - U \quad \Rightarrow \quad \boxed{F = U - TS}$$

then $dF = -S\,dT - P\,dV$, so $F = F(T,V)$

For a system in equilibrium with a heat bath (aka thermal reservoir), maximizing the entropy of (system + heat bath) is equivalent to minimizing the free energy of the system.

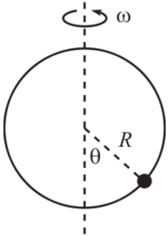
F = "Helmholtz free energy"

is Legendre transform of internal energy U.

Question:

- When is $\mathcal{H}$ conserved (i.e. a constant of the motion)?
- When does $\mathcal{H}$ equal the total energy?
- Notice that these are two different questions.

Morin 15.11. A bead is free to slide along a frictionless hoop of radius R. The hoop is forced to rotate with constant angular speed ω around a vertical diameter. Find $\mathcal{H}$ in terms of θ and p_θ, then write down Hamilton's equations. Is $\mathcal{H}$ the energy? Is $\mathcal{H}$ conserved?



$\phi = \omega t \rightarrow \dot\phi = \omega$ $\quad$ $(x,y) = R(\cos\omega t, \sin\omega t)$ which depends on time explicitly

$T = \frac{m}{2}\left(R^2\dot\theta^2 + (R\sin\theta)^2\omega^2\right)$ $\qquad U = -mgR\cos\theta$

$$\mathcal{L} = \tfrac{1}{2}mR^2\dot\theta^2 + \tfrac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

$$p = \frac{\partial\mathcal{L}}{\partial\dot\theta} = mR^2\dot\theta \Rightarrow \dot\theta = \frac{p}{mR^2}$$

$$T = \tfrac{1}{2}mR^2\left(\frac{p}{mR^2}\right)^2 + \tfrac{1}{2}mR^2\omega^2\sin^2\theta = \frac{p^2}{2mR^2} + \tfrac{1}{2}mR^2\omega^2\sin^2\theta$$

$$\mathcal{H} = p\dot\theta - \mathcal{L} = \frac{p^2}{mR^2} - \frac{p^2}{2mR^2} - \tfrac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

$$\mathcal{H} = \frac{p^2}{2mR^2} - \tfrac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

$$\dot\theta = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{mR^2} \quad \Bigg\} \quad \dot p = -\frac{\partial\mathcal{H}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{mR^2\omega^2}{2}\sin^2\theta + mgR\cos\theta\right)$$

$$\dot p = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

$$\rightarrow \ddot\theta = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

What is the energy? $\frac{p^2}{2mR^2} + \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta \neq \mathcal{H}$

It seems that $\mathcal{H} = \text{energy} - mR^2\omega^2\sin^2\theta$

$\frac{\partial\mathcal{L}}{\partial t} = -\frac{\partial\mathcal{H}}{\partial t} = 0 \Rightarrow \mathcal{H}$ is conserved! (but it's not the energy)

$\rightarrow$ energy $=$ const. $+ mR^2\omega^2\sin^2\theta$

(We can show by explicit calculation that $\frac{dH}{dt}=0$)

$$mR^2\ddot{\theta}\boxed{\dot{\theta}} = mR^2\omega^2\sin\theta\cos\theta\boxed{\dot{\theta}} - mgR\sin\theta\boxed{\dot{\theta}}$$

$$\left\{\begin{array}{l} d\theta = \dot{\theta}\,dt \\ d\dot{\theta} = \ddot{\theta}\,dt \\ \frac{1}{2}d(\dot{\theta}^2) = \dot{\theta}\ddot{\theta}\,dt \end{array}\right.$$

$$\frac{d}{dt}\left(\frac{1}{2}mR^2\dot{\theta}^2\right) = mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta} - mgR\sin\theta\,\dot{\theta}$$

$$= \frac{d}{dt}\left(\frac{p^2}{2mR^2}\right)$$

$$\rightarrow \frac{dH}{dt} = \frac{d}{dt}\left(\frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta\right)$$

$$= \frac{d}{dt}\left(\frac{p^2}{2mR^2}\right) - mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta} + mgR\sin\theta\,\dot{\theta} = 0 \quad ✓$$

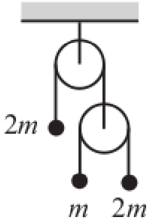
The quantity $mR^2\omega^2\sin^2\theta$ turns out to be the work done on the bead by the force of constraint that enforces $\phi = \omega t$.

$$\tau_z = \frac{dL_z}{dt} = \frac{d}{dt}\left(m(R\sin\theta)^2\omega\right) = 2mR^2\omega\sin\theta\cos\theta\,\dot{\theta}$$

Power delivered by this torque $= \frac{d(\text{work})}{dt} = \omega\tau_z = 2mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta}$

$$= \frac{d}{dt}\left(mR^2\omega^2\sin^2\theta\right)$$

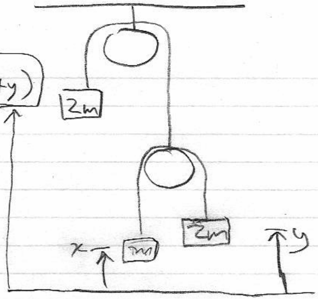
Morin 15.7. Consider the Atwood machine shown in the figure. Let x and y be the vertical positions of the middle mass and right mass, respectively, with upward taken to be positive. Find $\mathcal{H}$ in terms of x and y and their conjugate momenta, then write down the four Hamilton's equations.



(The solution to this problem turns out to be not especially illuminating. But it does illustrate how tedious the Hamiltonian method can be for solving problems that are straightforward using the Lagrangian method.)

Morin writes: "If you want to demonstrate how the Hamiltonian method can be monumentally more cumbersome than the Lagrangian method, you can try to solve this problem in the case of three general masses, m_1, m_2, m_3." ♣

$$\boxed{\text{const.} - \tfrac{1}{2}(x+y)}$$



$$U = mgx + 2mgy - 2mg\,\frac{x+y}{2}$$

$$= mgx + 2mgy - mg(x+y)$$

$$U = mgy$$

$$T = \frac{m}{2}\dot{x}^2 + m\dot{y}^2 + \left(\frac{2m}{2}\right)\left(\frac{\dot{x}+\dot{y}}{2}\right)^2$$

$$= \frac{m}{2}\dot{x}^2 + m\dot{y}^2 + \frac{m}{4}\left(\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2\right) = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2$$

$$\mathcal{L} = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2 - mgy$$

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{3}{2}m\dot{x} + \frac{1}{2}m\dot{y}$$

$$p_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{1}{2}m\dot{x} + \frac{5}{2}m\dot{y}$$

$$p_x - 3p_y = \left(\frac{1}{2} - \frac{15}{2}\right)m\dot{y} = -7m\dot{y} \Rightarrow \dot{y} = \frac{3p_y - p_x}{7m}$$

$$-5p_x + p_y = \left(-\frac{15}{2} + \frac{1}{2}\right)m\dot{x} = -7m\dot{x} \Rightarrow \dot{x} = \frac{5p_x - p_y}{7m}$$

$$\mathcal{L} = \frac{3}{4} m\dot{x}^2 + \frac{1}{2} m\dot{x}\dot{y} + \frac{5}{4} m\dot{y}^2 - mgy$$

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$$T = \frac{3m}{4}\left(\frac{5p_x - p_y}{7m}\right)^2 + \frac{m}{2}\left(\frac{5p_x - p_y}{7m}\right)\left(\frac{3p_y - p_x}{7m}\right) + \frac{5m}{4}\left(\frac{3p_y - p_x}{7m}\right)^2$$

$$T = \frac{1}{14m}\left(5p_x^2 - 2p_x p_y + 3p_y^2\right) \qquad U = mgy$$

$$H = T + U = \left(\frac{1}{14m}\right)\left(5p_x^2 - 2p_x p_y + 3p_y^2\right) + mgy$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{14m}\left(10p_x - 2p_y\right) = \frac{5p_x - p_y}{7m}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{14m}\left(-2p_x + 6p_y\right) = \frac{3p_y - p_x}{7m}$$

we knew that already

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad \ddot{x} = -\frac{\dot{p}_y}{7m} = \boxed{\frac{g}{7} = \ddot{x}}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -mg \qquad \ddot{y} = \frac{3\dot{p}_y}{7m} = \boxed{-\frac{3}{7}g = \ddot{y}} \qquad \frac{\ddot{x} + \ddot{y}}{2} = -\frac{g}{7}$$

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One answer: In the classical approximation, $S \gg \hbar$ so the phase contribution is very large. So the action S is an extremum for the special path $\overline{x}(t)$. All the contributions for the paths in this region are nearly in phase at $S_{\text{classical}}/\hbar$ and do not cancel out. So, in the classical limit we only need to consider the paths in the vicinity of $\overline{x}(t)$ as giving important contributions [to the quantum-mechanical amplitude]. So in this way, the classical laws of motion arise from the quantum laws.

In the reading, Feynman argued that the classical action

$$S_{\mathrm{cl}} = \int_{t_i}^{t_f} \mathcal{L}(\dot{x}(t), x(t), t)\, \mathrm{d}t$$

is proportional to the trajectory's quantum-mechanical phase:

$$\text{phase} = S_{\mathrm{cl}}/\hbar$$

Many of you noticed that Feynman's Problems 2-4 and 2-5 suggest a way to prove, using calculus of variations, that

$$(p)_{x=x_f} = \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_{x=x_f} = +\frac{\partial S_{\mathrm{cl}}}{\partial x_f} \qquad \text{and} \qquad E = \mathcal{H} = -\frac{\partial S_{\mathrm{cl}}}{\partial t_f}$$

Here's another route to that result: Remember that

$$\mathcal{H} = p\dot{x} - \mathcal{L} \qquad \Rightarrow \qquad \mathcal{L} = p\dot{x} - \mathcal{H}$$

So we can rewrite the classical action as

$$S_{\mathrm{cl}} = \int_{t_i}^{t_f} (p\dot{x} - \mathcal{H})\, \mathrm{d}t = \int_{t_i}^{t_f} p\dot{x}\, \mathrm{d}t - \int_{t_i}^{t_f} \mathcal{H}\, \mathrm{d}t = \int_{x_i}^{x_f} p\, \mathrm{d}x - \int_{t_i}^{t_f} \mathcal{H}\, \mathrm{d}t$$

$$S = \int_{t_i}^{t} \mathcal{L}\,\mathrm{d}t = \int_{t_i}^{t} (p\dot{x} - \mathcal{H})\,\mathrm{d}t = \int_{x_i}^{x} p\,\mathrm{d}x - \int_{t_i}^{t} \mathcal{H}\,\mathrm{d}t$$

Therefore,

$$\left(\frac{\partial S}{\partial t}\right)_{\text{fixed x}} = -\mathcal{H} \qquad \text{and} \qquad \left(\frac{\partial S}{\partial x}\right)_{\text{fixed t}} = p$$

$\partial S/\partial t + \mathcal{H} = 0$ is the "Hamilton-Jacobi equation." If we plug in

$$\mathcal{H} = \frac{p^2}{2m} + U(x)$$

we can write this differential equation for the classical action:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x) = 0$$

or in three dimensions,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + U(\boldsymbol{r}) = 0$$

What is this telling us?

$$\frac{\partial S}{\partial t} = -E \qquad \frac{\partial S}{\partial x} = p_x \qquad \frac{\partial S}{\partial y} = p_y \qquad \frac{\partial S}{\partial z} = p_z \qquad \nabla S = \boldsymbol{p}$$

For constant energy, an action of the form

$$S(\boldsymbol{r}, t) = \boldsymbol{p} \cdot \boldsymbol{r} - Et$$

satisfies these equations. Notice that momentum $\boldsymbol{p}$ is $\perp$ to surface of constant S. Near the classical path, moving $\perp$ to the trajectory does not change the action — as we expect from the "principle of stationary action."

In Physics 250, you may have described "matter waves" using the de Broglie relations $\boldsymbol{p} = \hbar \boldsymbol{k}$ and $E = \hbar\omega$. This suggests

$$S(\boldsymbol{r}, t)/\hbar = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$$

which describes the phase of a plane wave.

Meanwhile, the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + U(\boldsymbol{r}) = 0$$

is starting to seem vaguely similar to Schrödinger's equation:

$$i\hbar\,\frac{\partial \psi(\boldsymbol{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{r})\right)\,\psi(\boldsymbol{r},t)$$

Let's try plugging (into Schrödinger) a wavefunction

$$\psi(x,t) = \psi_0(x,t)\,e^{i\Sigma(x,t)/\hbar}$$

where $\psi_0(x,t)$ and $\Sigma(x,t)$ are real (i.e. not complex) functions. So $|\psi_0|^2$ tells us about probability, and $\Sigma/\hbar$ tells us about phase.

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi_0}{\partial t}\,e^{i\Sigma/\hbar} + \left(\frac{i}{\hbar}\frac{\partial \Sigma}{\partial t}\right)\psi_0\,e^{i\Sigma/\hbar}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi_0}{\partial x^2}e^{i\Sigma/\hbar} + \frac{2i}{\hbar}\frac{\partial \Sigma}{\partial x}\frac{\partial \psi_0}{\partial x}e^{i\Sigma/\hbar} + \frac{i}{\hbar}\frac{\partial^2 \Sigma}{\partial x^2}\psi_0 e^{i\Sigma/\hbar} - \frac{1}{\hbar^2}\left(\frac{\partial \Sigma}{\partial x}\right)^2\psi_0 e^{i\Sigma/\hbar}$$

Plugging in and canceling common factor $\psi_0 e^{i\Sigma/\hbar}$ gives real part

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{2m}\left(\frac{\partial \Sigma}{\partial x}\right)^2 + U = \frac{\hbar^2}{2m}\frac{1}{\psi_0}\frac{\partial^2 \psi_0}{\partial x^2}$$

which equals the Hamilton-Jacobi equation, "in $\hbar \to 0$ limit." So evidently in some classical limit, the phase Σ of Schrödinger's $\psi(x,t)$ satisfies the same diffeq. as does the classical action S.

The imaginary part gives (skip the math here)

$$\frac{\partial \psi_0}{\partial t} + \frac{1}{m}\frac{\partial \Sigma}{\partial x}\frac{\partial \psi_0}{\partial x} + \frac{1}{2m}\psi_0 \frac{\partial^2 \Sigma}{\partial x^2} = 0$$

which can be turned into (multiply by $2\psi_0$, use $\partial\Sigma/\partial x \to p$ if $\Sigma \to S$)

$$\frac{\partial}{\partial t}(\psi_0^2) + \frac{\partial}{\partial x}(\psi_0^2 \frac{1}{m}\frac{\partial \Sigma}{\partial x}) = 0$$

$$\frac{\partial}{\partial t}(\psi_0^2) + \nabla \cdot (\psi_0^2\, \boldsymbol{v}) = 0$$

which is (Taylor 16.130) just the continuity equation expressing conservation of probability (ψ_0^2) as the particle travels.

The Schrödinger equation gives us

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{2m}\left(\frac{\partial \Sigma}{\partial x}\right)^2 + U = \frac{\hbar^2}{2m}\frac{1}{\psi_0}\frac{\partial^2 \psi_0}{\partial x^2}$$

while the classical Hamilton-Jacobi equation gave us

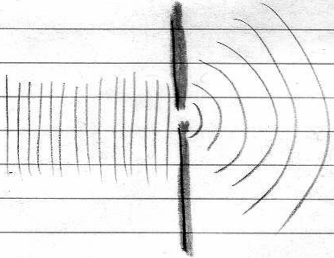
$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U = 0$$

What does it mean for $\frac{\hbar^2}{2m}\frac{1}{\psi_0}\frac{\partial^2 \psi_0}{\partial x^2}$ to be "small?" Consider a gaussian distribution $\psi_0(x) \sim e^{-x^2/2\sigma^2}$. Then

$$\left(\frac{1}{\psi_0}\frac{\partial^2 \psi_0}{\partial x^2}\right)_{x=0} = -\frac{1}{\sigma^2} \sim \frac{1}{L^2}$$

where L is the length over which the probability for finding the particle varies appreciably, e.g. slit size, or distance over which $U(x)$ varies considerably. Then classical limit means

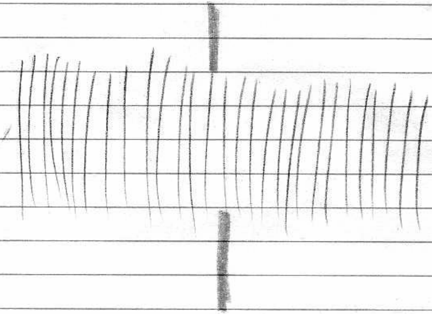
$$\frac{p^2}{2m} \gg \frac{\hbar^2}{2mL^2} \qquad\qquad p \gg \frac{\hbar}{L} \qquad\qquad L \gg \lambda_{\text{de Broglie}}$$



$L \sim \lambda$

(diffraction)

wave optics



$L \gg \lambda$

(ray optics is OK)

Physics 351 — Monday, April 23, 2018

- Turn in HW12. Last one! Hooray!
- Last day to turn in XC is Sunday, May 6 (three days after the exam). For the few people who did Perusall (sorry!), I will factor that in as XC.
- There is interest in forming study groups to help review or catch up on material from this course, as the semester winds down. Learning physics really is a lot more fun when it is done cooperatively. To try to faciliate this, I created a Canvas discussion area, but so far **only one person** has followed up.
- Don't forget the last two readings: Feynman/Hibbs (on Canvas) for today; and Feynman's two lectures on fluids, for Wednesday.
- I meant to finish Friday by saying that whereas Taylor's first example of a canonical transformation was, in effect, a 90° rotation in (p,q) space, his second example was effectively a Cartesian-to-polar transformation of (p,q) space.