Physics 351 review session — 2018-04-29

- Final exam: Thursday, May 3, 9am–11am, DRL A2. Covers chapters 7,9,10,13. One hand-written 3×5 card OK. Turn in your 3×5 card with your exam.
- HW5,7,9,10,11,12 all contain good exam-problem candidates. But HW10 was graded after last day of class, so HW10 problems are only fair game if they overlap with example 2015/2017 exams. Anything from 2015/2017 final exams is a good exam-problem candidate (to use or to adapt).
- Last day to turn in XC is Sunday, May 6 (3 days after exam).
- If you felt that you just could not get enough of Feynman's introduction to quantum ideas, feel free to read for XC (and summarize in writing) the first 3 or 4 chapters of Volume III (QM) of the Feynman Lectures on Physics.

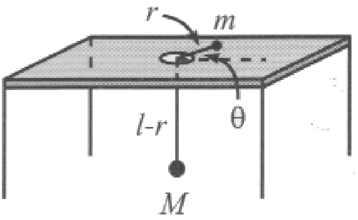
`http://positron.hep.upenn.edu/p351/files/exam2015.pdf`

`http://positron.hep.upenn.edu/p351/files/exam2017.pdf`

`http://positron.hep.upenn.edu/p351/files/exam2015_solns.pdf`

`http://positron.hep.upenn.edu/p351/files/exam2017_solns.pdf`

This was on the midterm. Let's redo it as a Hamiltonian problem.
A mass m is free to slide on a frictionless table and is connected, via a string that passes through a hole in the table, to a mass M that hangs below. Assume that M moves in a vertical line only, and assume that the string always remains taut. (a) Find the EOM for r and for θ as shown in the left figure below. (b) Under what condition does m undergo circular motion? (c) What is the frequency of small oscillations (in the variable r) about this circular motion (i.e. if the orbit is perturbed slightly w.r.t. the circular motion)?



http://positron.hep.upenn.edu/p351/files/midterm_2018_solns.pdf

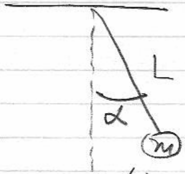
I have two example Coriolis problems here. One is the Foucault pendulum, which we solved in class on the day before spring break, using Lagrangian mechanics. We could either do the Foucault pendulum using Newtonian mechanics, or else do the "puck slides on large, perfectly level, frictionless sheet of ice" problem. Which do you prefer?

$$\underset{\sim}{\Omega} = \hat{z}\cos\theta + \hat{y}\sin\theta$$

$$\underset{\sim}{\Omega} \times \hat{x} = \Omega\cos\theta\,\hat{y} - \Omega\sin\theta\,\hat{z}$$

$$\underset{\sim}{\Omega} \times \hat{y} = -\Omega\cos\theta\,\hat{x}$$

$$\underset{\sim}{\Omega} \times \hat{z} = \Omega\sin\theta\,\hat{x}$$



$$U = mgL(1-\cos\alpha) \simeq mgL\frac{\alpha^2}{2}$$

$$\underset{\sim}{r} = x\hat{x} + y\hat{y} + R\hat{z}$$

$$\dot{\underset{\sim}{r}}_0 = \dot{\underset{\sim}{r}} + \underset{\sim}{\Omega} \times \underset{\sim}{r} = \dot{x}\hat{x} + \dot{y}\hat{y} + \Omega x(\hat{y}\cos\theta - \hat{z}\sin\theta) - \Omega\cos\theta\, y\hat{x} + \Omega\sin\theta\, R\hat{x}$$

$$\dot{\underset{\sim}{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

$$|\dot{\underset{\sim}{r}}_0|^2 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)^2 + (\dot{y} + x\Omega\cos\theta)^2 + (x\Omega\sin\theta)^2$$

(drop any Ω^2 terms)

$$\dot{\underset{\sim}{r}}_0 = (\dot{x} - y\Omega\cos\theta + R\Omega\sin\theta)\hat{x} + (\dot{y} + x\Omega\cos\theta)\hat{y} - x\Omega\sin\theta\,\hat{z}$$

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(drop any Ω^2 terms)

$$|\dot{\underset{\sim}{r}}_0|^2 = \dot{x}^2 - 2\dot{x}y\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta + \dot{y}^2 + 2\dot{y}x\Omega\cos\theta$$

$$= \dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta$$

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right]$$

$$U \simeq mg\frac{(\alpha L)^2}{2L} = mg\frac{(x^2+y^2)}{2L} = \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}(x^2+y^2)$$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + 2(\dot{y}x - \dot{x}y)\Omega\cos\theta + 2\dot{x}R\Omega\sin\theta\right] - \frac{mg}{2L}(x^2+y^2)$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) \Rightarrow m\dot{y}\Omega\cos\theta - \frac{mgx}{L} = \frac{d}{dt}\left(m\dot{x} - my\Omega\cos\theta + 2R\Omega\sin\theta\right)$$

$$\ddot{x} - \dot{y}\Omega\cos\theta = \dot{y}\Omega\cos\theta - \frac{gx}{L} \Rightarrow \ddot{x} = -\frac{g}{L}x + 2\Omega\cos\theta\,\dot{y}$$

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) \Rightarrow -m\dot{x}\Omega\cos\theta - \frac{mgy}{L} = \frac{d}{dt}\left(m\dot{y} + mx\Omega\cos\theta\right) = m\ddot{y} + m\dot{x}\Omega\cos\theta$$

$$\ddot{y} = -\frac{g}{L}y - 2\Omega\cos\theta\,\dot{x}$$

let $\Omega_y = \Omega\cos\theta$

$$\left.\begin{aligned} \ddot{x} &= -\frac{g}{L}x + 2\Omega_y\dot{y} \\ \ddot{y} &= -\frac{g}{L}y - 2\Omega_y\dot{x} \end{aligned}\right.$$

$$\text{try } \eta \equiv x + iy = Ae^{iBt} \Rightarrow \dot{\eta} = iB\eta, \quad \ddot{\eta} = -B^2\eta$$

$$\ddot{x} = -\omega_0^2 x + 2\Omega_y \dot{y} \qquad \ddot{y} = -\omega_0^2 y - 2\Omega_y \dot{x}$$

$$\ddot{x} + i\ddot{y} = -\omega_0^2(x+iy) + 2\Omega_y(\dot{y} - i\dot{x}) = -\omega_0^2(x+iy) - 2i\Omega_y(\dot{x} + i\dot{y})$$

$$\ddot{\eta} = -\omega_0^2 \eta - 2i\Omega_y \dot{\eta} \Rightarrow -B^2 = -\omega_0^2 - (2i\Omega_y)(iB)$$

$$-B^2 = -\omega_0^2 + 2\Omega_y B \Rightarrow B^2 + 2B\Omega_y - \omega_0^2 = 0$$

$$B = \frac{1}{2}\left(-2\Omega_y \pm \sqrt{4\Omega_y^2 + 4\omega_0^2}\right) = -\Omega_y \pm \sqrt{\Omega_y^2 + \omega_0^2} \simeq -\Omega_y \pm \omega_0$$

$$x + iy = A_1 e^{-i\Omega_y t} e^{i\omega_0 t} + A_2 e^{-i\Omega_y t} e^{-i\omega_0 t}$$

$$= e^{-i\Omega_y t}\left(C_1 \cos\omega_0 t + \cancel{C_2 \sin\omega_0 t}\right)$$

$$x(t) = C\cos(\Omega_y t)\cos(\omega_0 t)$$

$$y(t) = -C\sin(\Omega_y t)\cos(\omega_0 t)$$

Oops — of course I meant to write Omega_z, not Omega_y.

A puck slides with speed v on frictionless ice. The surface is "level" in the sense that it is orthogonal to the effective (gravitational + centrifugal) $\boldsymbol{g}$ at all points. Show that the puck moves in a circle, as seen in Earth's rotating frame. (Assume that v is small enough that the radius of the circle is much smaller than the radius of Earth, so that the colatitude θ is essentially constant throughout the motion.) What is the radius of the circle? What is the frequency of the motion?

Let $\hat{x}$ point east, $\hat{y}$ point north, $\hat{z}$ point "up" so that $\vec{g} = -g\hat{z}$.

Earth's rotation vector is $\vec{\Omega}$ where $|\Omega| \simeq \frac{2\pi}{86400\,s} \simeq 7.3 \times 10^{-5}\, s^{-1}$, and

$$\hat{\Omega} = \cos\theta\, \hat{z} + \sin\theta\, \hat{y}$$

$$\vec{F}_{coriolis} = 2m\vec{v} \times \vec{\Omega}$$

$$= 2m\left[(v_y\Omega_z - \underset{0}{v_z}\Omega_y)\hat{x} + (\underset{0}{v_z}\Omega_x - v_x\Omega_z)\hat{y} + (v_x\Omega_y - v_y\underset{0}{\Omega_x})\hat{z}\right]$$

Ignore F_z, since we're on frictionless ice, and since we'll assume $|v_x\Omega_y| \ll g$, so puck will not go airborne.

$$m\ddot{x} = 2m v_y \Omega_z = 2m\Omega\cos\theta\, v_y = m\dot{v}_x$$
$$m\ddot{y} = -2m v_x \Omega_z = -2m\Omega\cos\theta\, v_x = m\dot{v}_y$$

$$\dot{v}_x = (2\Omega\cos\theta)\, v_y \quad , \quad \dot{v}_y = -(2\Omega)\cos\theta\, v_x$$

let $\eta = v_x + i v_y = A e^{i\omega t} \longrightarrow \dot{\eta} = i\omega\eta$

$$\dot{v}_x + i\dot{v}_y = i\omega v_x - \omega v_y \longrightarrow \dot{v}_x = -\omega v_y \,,\ \dot{v}_y = \omega v_x$$
$$\Longrightarrow \boxed{\omega = -2\Omega\cos\theta}$$

$$x + iy = \frac{A}{i\omega}(\cos\omega t + i\sin\omega t) = \frac{|A|e^{i\delta}}{i\omega}(\cos\omega t + i\sin\omega t)$$

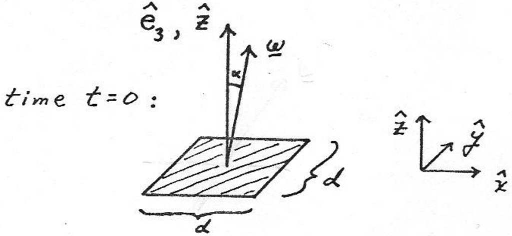
$$x = \frac{|A|}{\omega}\sin(\omega t+\delta) \qquad y = -\frac{|A|}{\omega}\cos(\omega t+\delta)$$

which describes a circle of radius $\frac{|A|}{|\omega|} = \boxed{\frac{v}{2\Omega\cos\theta}}$
and angular frequency $2\Omega\cos\theta$

If $\theta = 45^\circ$, $v = 1\frac{m}{s}$, then $R \simeq 10\,\text{km}$! (small effect!)

From the final exam for the course I took, fall 1990. (This turns out to be the same problem as appears in Feynman's story of the cafeteria plate that wobbles as it flies through the air.)

An infinitely thin, uniform, square plate of mass m and side d is allowed to undergo rotation. At time $t = 0$, the normal to the plate, $\hat{e}_3$, is aligned with $\hat{z}$, but the angular velocity vector $\underline{\omega}$ deviates from $\hat{z}$ by a small angle α. Work the entire problem to first order in α, i.e. drop terms of $O(\alpha^2)$ or higher.



(a) Show $I = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find I_0.

(b) Find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate.

(c) When is this maximum deviation first reached?

$$\boldsymbol{\omega} \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$$

symmetric top: $\lambda_1 = \lambda_2 \quad \Rightarrow \quad \boldsymbol{L} \;=\; \lambda_1 \omega_1 \hat{\boldsymbol{e}}_1 + \lambda_1 \omega_2 \hat{\boldsymbol{e}}_2 + \lambda_3 \omega_3 \hat{\boldsymbol{e}}_3$

$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3 \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3 - \omega_3 \hat{\boldsymbol{e}}_3$$

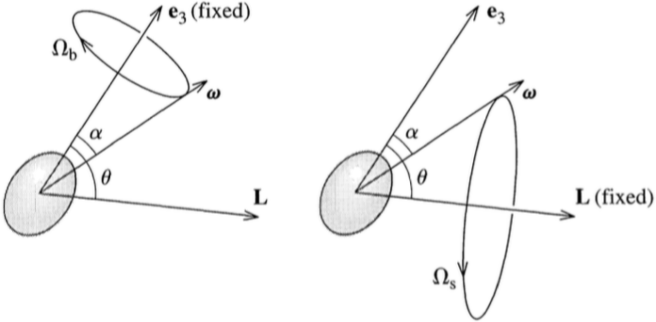
$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \boldsymbol{\omega} + \frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3$$

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \left(\frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \right) \hat{\boldsymbol{e}}_3 \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \, \hat{\boldsymbol{e}}_3$$

Last line proves that $\boldsymbol{\omega}$, $\boldsymbol{L}$, and $\hat{e}_3$ are coplanar (for $\lambda_1 = \lambda_2$).

Torque-free (10.94): $\boldsymbol{\omega} \;=\; \omega_0 \cos(\Omega_b t) \hat{\boldsymbol{e}}_1 - \omega_0 \sin(\Omega_b t) \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$

Key trick for understanding “space” and “body” cones: decompose $\boldsymbol{\omega}$ into one part that points along $\boldsymbol{L}$ and one part that points along (or opposite) $\hat{e}_3$. [Sign of Ω_b depends on λ_1 vs. λ_3 magnitudes.]



(a) Body frame (b) Space frame

Figure 10.9 An axially symmetric body (shown here as a prolate spheroid or "egg-shaped" solid) is rotating with angular velocity $\boldsymbol{\omega}$, not in the direction of any of the principal axes. **(a)** As seen in the body frame, both $\boldsymbol{\omega}$ and $\mathbf{L}$ precess about the symmetry axis, $\mathbf{e}_3$, with angular frequency Ω_b given by (10.93). **(b)** As seen in the space frame, $\mathbf{L}$ is fixed, and both $\boldsymbol{\omega}$ and $\mathbf{e}_3$ precess about $\mathbf{L}$ with frequency Ω_s given by (10.96).

Torque-free precession of axially symmetric ($\lambda_1 = \lambda_2$) rigid body

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3 \quad \text{with} \quad \Omega_b \;=\; \frac{\lambda_1 - \lambda_3}{\lambda_1}\,\omega_3$$

$$\boldsymbol{\omega} \;=\; \omega_0 \cos(\Omega_b t)\hat{\boldsymbol{e}}_1 - \omega_0 \sin(\Omega_b t)\hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$$

$\boldsymbol{\Omega}_{\text{space}} = \boldsymbol{L}/\lambda_1$ points along $\boldsymbol{L}$. Describes precession of $\boldsymbol{\omega}$ (and $\hat{\boldsymbol{e}}_3$) about $\boldsymbol{L}$ as seen in space frame.

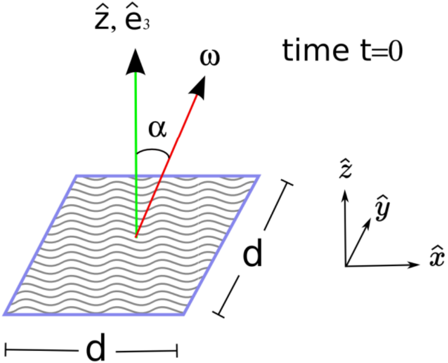
$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_3}{\mathrm{d}t} \;=\; \boldsymbol{\omega}\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1}\right)\times\hat{\boldsymbol{e}}_3 \;=\; \boldsymbol{\Omega}_{\text{space}}\times\hat{\boldsymbol{e}}_3$$

$\boldsymbol{\Omega}_{\text{body}} = -\Omega_b \hat{\boldsymbol{e}}_3$ points along $\hat{\boldsymbol{e}}_3$ if $\lambda_3 > \lambda_1$ (oblate, frisbee) and points opposite $\hat{\boldsymbol{e}}_3$ if $\lambda_3 < \lambda_1$ (prolate, US football). Describes precession of $\boldsymbol{\omega}$ (and $\boldsymbol{L}$) about $\hat{\boldsymbol{e}}_3$ as seen in body frame.

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{\text{body}} = -\boldsymbol{\omega}\times\boldsymbol{L} = -\left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\boldsymbol{L} = (-\Omega_b \hat{\boldsymbol{e}}_3)\times\boldsymbol{L} = \boldsymbol{\Omega}_{\text{body}}\times\boldsymbol{L}$$

$$\boldsymbol{\Omega}_{\text{space}} \;=\; \boldsymbol{\omega} \;+\; \boldsymbol{\Omega}_{\text{body}}$$

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(a) Show that $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0.

(b) Calculate $\boldsymbol{L}$ at $t = 0$.

(c) Sketch $\hat{e}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$.

(d) Draw/label “body cone” and “space cone” on your sketch.

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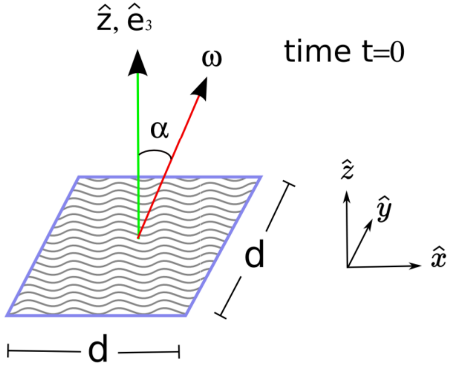

(e) Calculate precession frequencies Ω_{body} and Ω_{space}. Indicate directions of precession vectors $\boldsymbol{\Omega}_{\textbf{body}}$ and $\boldsymbol{\Omega}_{\textbf{space}}$ on drawing.

(f) You argue in HW that $\boldsymbol{\Omega}_{\textbf{space}} = \boldsymbol{\Omega}_{\textbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\textbf{space}}$ and $\boldsymbol{\Omega}_{\textbf{body}}$ that you calculate for $t = 0$.

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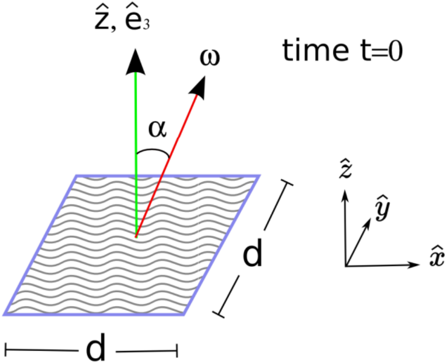

(g) Find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{e}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α.

(h) When is this maximum deviation first reached?

video: `https://www.youtube.com/watch?v=oH-dlrIFO10`

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