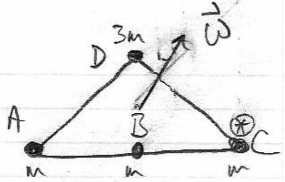
Physics 351 — Monday, April 6, 2020

- I'm Bill Ashmanskas (Bill, Dr Bill, Prof Bill, etc), filling in for Prof Liu today. I taught Phys 351 in 2015, 2017, 2018.
- Today's focus: Taylor §10.7 & §10.8.
- These slides will be on Canvas/files as `p351_notes_20200406.pdf`
- I pre-recorded a few short mini-lectures on Euler equations:
  `https://tinyurl.com/qoya2dz`
  `https://tinyurl.com/t76jlqz`
  `https://tinyurl.com/rrjdcza`
  `https://tinyurl.com/tuycjkf`
  `https://tinyurl.com/vhgt9ea`



$$\vec{V}_{\text{immediately after impact}} = \vec{V}_{cm} + \vec{\omega} \times \vec{r}$$

$$\vec{V}_{cm} = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_A = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (-2a, -a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{3}\right)$$

$$\vec{\omega} \times \vec{r}_B = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{1}{6}\right)$$

$$\vec{\omega} \times \vec{r}_C = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (+2a, -a, 0) = \frac{P}{m}\left(0, 0, -\frac{2}{3}\right)$$

$$\vec{\omega} \times \vec{r}_D = \frac{P}{ma}\left(\frac{1}{6}, \frac{1}{4}, 0\right) \times (0, +a, 0) = \frac{P}{m}\left(0, 0, \frac{1}{6}\right)$$

$$\Rightarrow \quad \vec{V}_A = \frac{P}{m}\left(0, 0, \frac{1}{6}\right) \qquad \vec{V}_C = \frac{P}{m}\left(0, 0, -\frac{5}{6}\right)$$

$$\vec{V}_B = \frac{P}{m}\left(0, 0, -\frac{1}{3}\right) \qquad \vec{V}_D = \frac{P}{m}(0, 0, 0)$$

In[9]:=

$$\{1/6,\ 1/4,\ 0\} \times \{-2,\ -1,\ 0\}$$

Out[9]= $\left\{0,\ 0,\ \frac{1}{3}\right\}$

In[10]:= $\{1/6,\ 1/4,\ 0\} \times \{0,\ -1,\ 0\}$

Out[10]=

$$\left\{0,\ 0,\ -\frac{1}{6}\right\}$$

In[11]:= $\{1/6,\ 1/4,\ 0\} \times \{+2,\ -1,\ 0\}$

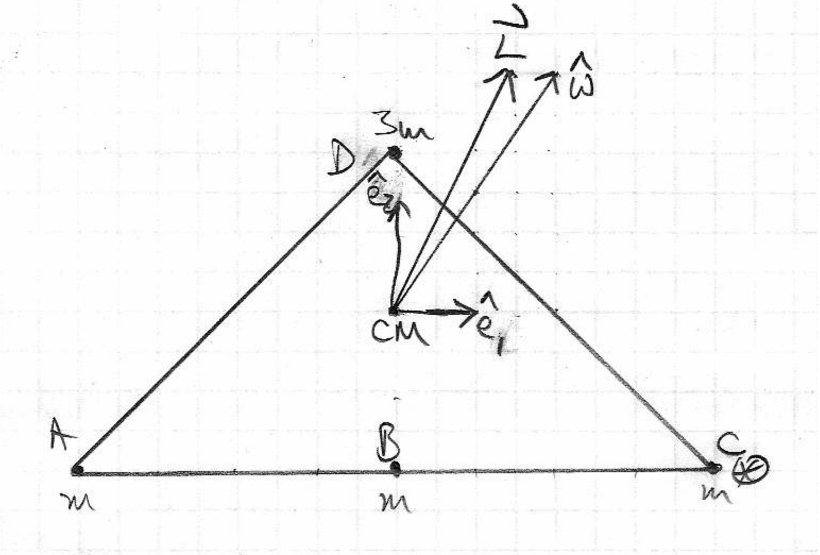
Out[11]=

$$\left\{0,\ 0,\ -\frac{2}{3}\right\}$$

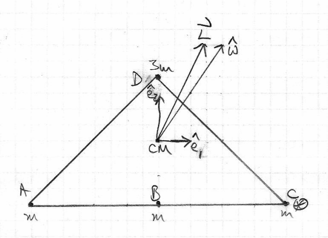
In[12]:= $\{1/6,\ 1/4,\ 0\} \times \{0,\ +1,\ 0\}$

Out[12]=

$$\left\{0,\ 0,\ \frac{1}{6}\right\}$$



What will the subsequently happen to $\boldsymbol{V}_{\mathrm{cm}}$? To $\boldsymbol{L}$? To $\boldsymbol{\omega}$? To the orientations of the principal axes? With no applied torque, how does $\boldsymbol{\omega}$ evolve in time?



$\lambda_1 = 6ma^2,\ \lambda_2 = 8ma^2,\ \lambda_3 = 14ma^2.$

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1, 2, 0)$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3\omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Try other initial ω vectors:

https://www.youtube.com/watch?v=dVhGyxkBKzI
https://www.youtube.com/watch?v=4Ntgvun8GuY
https://www.youtube.com/watch?v=YKSEu_c3YdY

I derive this more understandably in this mini-lecture video: https://tinyurl.com/qoya2dz

$$\underset{\sim}{\Gamma} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{space} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (\lambda_1 \dot{\omega}_1, \lambda_2 \dot{\omega}_2, \lambda_3 \dot{\omega}_3) + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + (\omega_1 L_2 - \omega_2 L_1)$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + \omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1 \omega_2$$

$$\lambda_3 \dot{\omega}_3 = \Gamma_3 + (\lambda_1 - \lambda_2)\omega_1 \omega_2$$

$$\lambda_1 \dot{\omega}_1 = \Gamma_1 + (\lambda_2 - \lambda_3)\omega_2 \omega_3$$

$$\lambda_2 \dot{\omega}_2 = \Gamma_2 + (\lambda_3 - \lambda_1)\omega_3 \omega_1$$

I derive this more understandably in this mini-lecture video:
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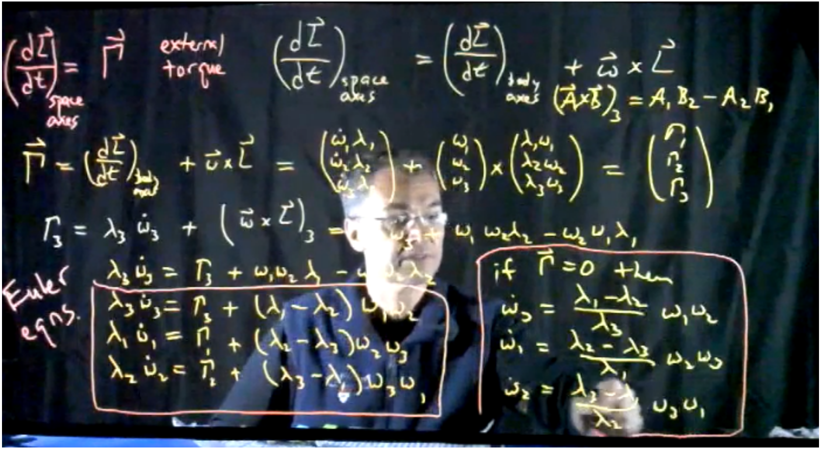


(For zero torque)

$$0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{space}\\\text{axes}}} = \left(\frac{d\vec{L}}{dt}\right)_{\substack{\text{body}\\\text{axes}}} + \vec{\omega}\times\vec{L}$$

$$(\dot{\omega}_1\lambda_1, \dot{\omega}_2\lambda_2, \dot{\omega}_3\lambda_3) = -\vec{\omega}\times(\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3)$$

$$= -(\omega_2\omega_3\lambda_3 - \omega_3\omega_2\lambda_2, \omega_3\omega_1\lambda_1 - \omega_1\omega_3\lambda_3, \omega_1\omega_2\lambda_2 - \omega_2\omega_1\lambda_1)$$

$$= (\omega_2\omega_3(\lambda_2-\lambda_3), \omega_1\omega_3(\lambda_3-\lambda_1), \omega_1\omega_2(\lambda_1-\lambda_2))$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{(\lambda_2-\lambda_3)}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1\omega_2\frac{\lambda_1-\lambda_2}{\lambda_3}$$

$$\dot{\omega}_2 = \omega_1\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2}$$

It's fun to consider e.g. $\lambda_3 > \lambda_2 > \lambda_1$ for tossed book.

I work through this (and the next page) more understandably in this mini-lecture video: `https://tinyurl.com/t76jlqz`

$\lambda_3 > \lambda_2 > \lambda_1$. Start out e.g. about $\hat{e}_3$, with ω_1 and ω_2 both $\ll \omega_3$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2-\lambda_3}{\lambda_1}$$

$$\ddot{\omega}_1 \simeq \dot{\omega}_2\omega_3 \frac{\lambda_2-\lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_1\omega_3 \frac{\lambda_3-\lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \underbrace{\omega_1\omega_2}_{\text{"small} \times \text{small"}} \frac{\lambda_1-\lambda_2}{\lambda_3}$$

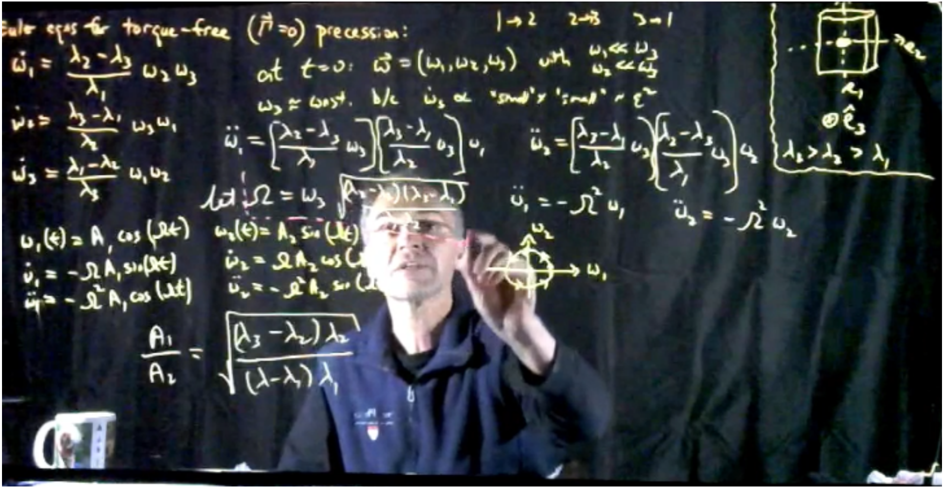
$\Rightarrow \omega_3 \approx$ constant (varies very slowly)

$$\ddot{\omega}_1 = \omega_1\left(\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2}\right)\left(\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1}\right) = -\omega_1\left(\omega_3^2\,\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{\lambda_1\lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq \dot{\omega}_1\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2} = \omega_2\left(\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1}\right)\left(\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2}\right)$$

$$\ddot{\omega}_2 \simeq -\omega_2\left(\omega_3^2\,\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\right) \equiv -\Omega^2\omega_2$$

I work through this (and the next page) more understandably in this mini-lecture video: `https://tinyurl.com/t76jlqz`



If you look down the $\hat{e}_3$ axis, you'll see the tip of $\boldsymbol{\omega}$ tracing out an ellipse whose ratio of axis lengths is $\sqrt{\frac{(\lambda_3-\lambda_2)\lambda_2}{(\lambda_3-\lambda_1)\lambda_1}}$.

$$\lambda_1\dot{\omega}_1 + \left[(\omega_1,\omega_2,\omega_3)\times(\lambda_1\omega_1,\lambda_2\omega_2,\lambda_3\omega_3)\right]_1 = 0$$

$$\lambda_1\dot{\omega}_1 + (\omega_2\lambda_3\omega_3 - \omega_3\lambda_2\omega_2) = 0$$

$$\lambda_1\dot{\omega}_1 + \omega_2\omega_3(\lambda_3-\lambda_2) = 0 \qquad (-\Omega^2 \quad)$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1} \rightarrow \ddot{\omega}_1 = \dot{\omega}_2\left(\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1}\right) = \omega_1\left(\omega_3^2\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\right)$$

$$\dot{\omega}_2 = \omega_3\omega_1\frac{\lambda_3-\lambda_1}{\lambda_2} \rightarrow \ddot{\omega}_2 = \dot{\omega}_1\left(\omega_3\frac{\lambda_3-\lambda_1}{\lambda_2}\right) = \omega_2\left(\omega_3^2\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\right)$$

$$\omega_1 = A\cos\Omega t \rightarrow \dot{\omega}_1 = -\Omega A\sin\Omega t = -\frac{\Omega A}{B}\omega_2 \longrightarrow \omega_3\frac{\lambda_3-\lambda_2}{\lambda_1} = \frac{\Omega A}{B}$$

$$\omega_2 = B\sin\Omega t \rightarrow \dot{\omega}_2 = \Omega B\cos\Omega t = \frac{\Omega B}{A}\omega_1 \longrightarrow \omega_3\frac{\lambda_3-\lambda_1}{\lambda_2} = \frac{\Omega B}{A}$$

$$\frac{A^2}{B^2} = \frac{(\lambda_3-\lambda_2)\lambda_2}{(\lambda_3-\lambda_1)\lambda_1}$$

I work through this (and the next page) more understandably in this mini-lecture video: https://tinyurl.com/rrjdcza

Start out about $\hat{e}_1$ $(\lambda_1 < \lambda_2 < \lambda_3)$, so ω_2 and ω_3 both $\ll \omega_1$ initially.

$$\dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \sim (\text{small})^2$$

$$\ddot{\omega}_2 \simeq \omega_1 \dot{\omega}_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} = -\omega_2 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) = -\Omega^2 \omega_2$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_3 \simeq \omega_1 \dot{\omega}_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = -\omega_3 \left(\omega_1^2 \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}{\lambda_2 \lambda_3} \right) \equiv -\Omega^2 \omega_3$$

Start out about $\hat{e}_2$, so ω_1 and $\omega_3 \ll \omega_2$ initially.

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_2 = \omega_1 \omega_3 \frac{\lambda_3 - \lambda_1}{\lambda_2} \sim (\text{small})^2$$

$$\ddot{\omega}_1 \simeq \omega_2 \dot{\omega}_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \qquad \dot{\omega}_3 = \omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \qquad \ddot{\omega}_3 \simeq \dot{\omega}_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$$\ddot{\omega}_1 \simeq \omega_2 \left(\omega_1 \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} \right) \frac{\lambda_2 - \lambda_3}{\lambda_1} = + \omega_1 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

$$\ddot{\omega}_3 \simeq \left(\omega_2 \omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) \omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3} = + \omega_3 \left(\omega_2^2 \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_3} \right)$$

$\Rightarrow$ exponential growth of ω_1, ω_3

$\longrightarrow$ initial motion about $\hat{e}_2$ won't stay about $\hat{e}_2$

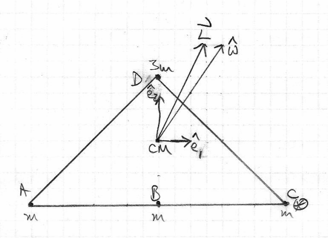
Video from two 2015 students traveling back from spring break:
https://www.youtube.com/watch?v=bVpPp1e_lZ4

Astronaut version:
https://youtu.be/fPI-rSwAQNg

Cosmonaut version (!): Dzhanibekov effect
https://youtu.be/dL6Pt1O_gSE

https://www.youtube.com/watch?v=BGRWg4aV2mw

Someone's quasi-intuitive explanation:
http://mathoverflow.net/questions/81960/
the-dzhanibekov-effect-an-exercise-in-mechanics-or-fiction-explain-mathemat



$\lambda_1 = 6ma^2$, $\lambda_2 = 8ma^2$, $\lambda_3 = 14ma^2$.

Space and body axes coincide at $t = 0$.

$\boldsymbol{\omega}_0 = \frac{P}{ma}(\frac{1}{6}, \frac{1}{4}, 0)$. $\boldsymbol{L} \equiv aP(1, 2, 0)$.

$$\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$$

$$\dot{\omega}_2 = \omega_3\omega_1 \frac{\lambda_3 - \lambda_1}{\lambda_2}$$

$$\dot{\omega}_3 = \omega_1\omega_2 \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.nb
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle.pdf
http://positron.hep.upenn.edu/p351/files/0327_strucktriangle_230.avi
https://www.youtube.com/watch?v=IMBRIyxDLss

Consider how you would go about calculating the (x, y, z) (space) positions of vertices A, C, D vs. time. I did it by keeping track of the (x, y, z) coordinates of the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ as a function of time.

(Neglecting $\vec{v}_{cm}$, as I did in animation)

$\vec{r}_A = -2a\hat{e}_1 - a\hat{e}_2 \qquad \vec{r}_C = +2a\hat{e}_1 - a\hat{e}_2$

$\vec{r}_D = +a\hat{e}_2$

Initially, $\hat{e}_1 = (1,0,0) \quad \hat{e}_2 = (0,1,0) \quad \hat{e}_3 = (0,0,1)$

We are given initial $\underset{\sim}{\omega}_{body}$.

$\underset{\sim}{\omega}_{space} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$

$\left(\frac{d\hat{e}_1}{dt}\right)_{space} = \underset{\sim}{\omega}_{space} \times \hat{e}_1 \quad \ldots\ldots \quad \left(\frac{d\hat{e}_3}{dt}\right)_{space} = \underset{\sim}{\omega}_{space} \times \hat{e}_3$

Lets me update $\hat{e}_1, \hat{e}_2, \hat{e}_3$ in x, y, z (space) coordinates

Then $\dot{\omega}_1 = \omega_2\omega_3 \frac{\lambda_2 - \lambda_3}{\lambda_1}$ etc.

lets me update $\underset{\sim}{\omega}_{body}$. Iterate for each frame of animation.

Torque-free precession of symmetric top:

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\lambda_3 \dot{\omega}_3 = \Gamma_3$ $(I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 = \text{const.}$ Suppose $\underset{\sim}{\Gamma} = 0$.

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\omega_2\omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\omega_3\omega_1 \end{aligned}\right\} \text{let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda}\omega_3$$

$$\dot{\omega}_1 = \Omega\,\omega_2 \,, \quad \dot{\omega}_2 = -\Omega\,\omega_1$$

$$\longrightarrow \begin{aligned} \omega_1 &= \omega_0 \cos\Omega t \\ \omega_2 &= -\omega_0 \sin\Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{\omega}$ precesses about $\hat{e}_3$ with frequency Ω.
As seen from the body frame, what does $\boldsymbol{L}$ do?

What does the situation look like from the space frame?

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\quad \lambda_3 \dot{\omega}_3 = \Gamma_3 \quad (I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 = \text{const.}$ Suppose $\underset{\sim}{\Gamma} = 0.$

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\omega_2\omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\omega_3\omega_1 \end{aligned}\right\} \text{ let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda}\omega_3$$

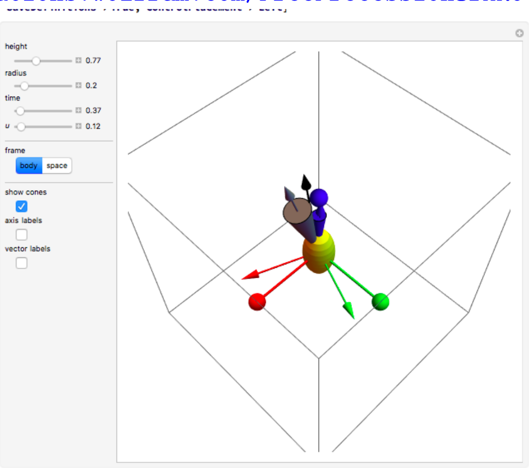
$$\dot{\omega}_1 = \Omega\,\omega_2 \,, \quad \dot{\omega}_2 = -\Omega\,\omega_1$$

$$\longrightarrow \quad \begin{aligned} \omega_1 &= \omega_0 \cos\Omega t \\ \omega_2 &= -\omega_0 \sin\Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ precess about (fixed) $\hat{e}_3$ with frequency $\Omega_b \equiv \Omega = \omega_3(\lambda - \lambda_3)/\lambda$, where $\lambda = \lambda_1 = \lambda_2$.

As seen from the space frame, $\hat{e}_3$ and $\boldsymbol{\omega}$ precess about (fixed) $\boldsymbol{L}$, at a frequency that takes some effort to calculate. (You'll calculate the space-frame precession frequency, Ω_s, on a future HW problem. It is much more involved than you might expect.)

http://demonstrations.wolfram.com/FreePrecessionOfARotatingRigidBody/

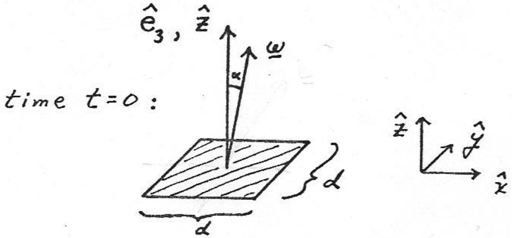


As seen from body frame, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ precess about (fixed) $\hat{e}_3$ with frequency $\Omega_b \equiv \Omega = \omega_3(\lambda - \lambda_3)/\lambda$, where $\lambda = \lambda_1 = \lambda_2$.

As seen from the space frame, $\hat{e}_3$ and $\boldsymbol{\omega}$ precess about (fixed) $\boldsymbol{L}$, at frequency $\Omega_s = L/\lambda_1$, which you'll prove in the HW.

From the final exam for the course I took, fall 1990. (This turns out to be the same problem as appears in Feynman's story of the cafeteria plate that wobbles as it flies through the air.)

An infinitely thin, uniform, square plate of mass m and side d is allowed to undergo rotation. At time $t = 0$, the normal to the plate, $\hat{e}_3$, is aligned with $\hat{z}$, but the angular velocity vector $\underline{\omega}$ deviates from $\hat{z}$ by a small angle α. Work the entire problem to first order in α, i.e. drop terms of $O(\alpha^2)$ or higher.



(a) Show $I = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find I_0.

(b) Find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate.

(c) When is this maximum deviation first reached?

If $\lambda_1 = \lambda_2 \equiv \lambda$ then $\quad \lambda_3 \dot{\omega}_3 = \Gamma_3 \quad (I_3 \alpha_3 = \tau_3)$

If $\Gamma_3 = 0$ then $\omega_3 = \text{const.}$ Suppose $\underset{\sim}{\Gamma} = 0$.

$$\left.\begin{aligned} \lambda \dot{\omega}_1 &= \cancel{\Gamma_1} + (\lambda - \lambda_3)\omega_2\omega_3 \\ \lambda \dot{\omega}_2 &= \cancel{\Gamma_2} + (\lambda_3 - \lambda)\omega_3\omega_1 \end{aligned}\right\} \text{let } \Omega \equiv \frac{\lambda - \lambda_3}{\lambda}\omega_3$$

$$\dot{\omega}_1 = \Omega\,\omega_2 \,, \quad \dot{\omega}_2 = -\Omega\,\omega_1$$

$$\longrightarrow \begin{aligned} \omega_1 &= \omega_0 \cos\Omega t \\ \omega_2 &= -\omega_0 \sin\Omega t \\ \omega_3 &\equiv \omega_3 \end{aligned}$$

As seen from body frame, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ precess about (fixed) $\hat{\boldsymbol{e}}_3$ with frequency $\Omega_b \equiv \Omega = \omega_3(\lambda - \lambda_3)/\lambda$, where $\lambda = \lambda_1 = \lambda_2$.

As seen from the space frame, $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{\omega}$ precess about (fixed) $\boldsymbol{L}$, at frequency $\Omega_s = L/\lambda_1$, which you'll prove in the HW.

`http://demonstrations.wolfram.com/FreePrecessionOfARotatingRigidBody/`

If $\underset{\sim}{\tau} = 0$ then

$$0 = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{space} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} + \underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\Rightarrow (\lambda_1 \dot{\omega}_1, \lambda_2 \dot{\omega}_2, \lambda_3 \dot{\omega}_3) = -\underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$\lambda_3 \dot{\omega}_3 = -(\omega_1 \omega_2 \lambda_2 - \omega_2 \omega_1 \lambda_1) = \omega_1 \omega_2 (\lambda_1 - \lambda_2)$$

$$\dot{\omega}_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3} \omega_1 \omega_2$$

$$\dot{\omega}_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1} \omega_2 \omega_3$$

$$\dot{\omega}_2 = \frac{\lambda_3 - \lambda_1}{\lambda_2} \omega_3 \omega_1$$

If $\lambda_1 = \lambda_2$ then $\dot{\omega}_3 = 0$

$$\dot{\omega}_1 = \boxed{\frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3} \omega_2 \equiv \Omega_b \omega_2$$

(Taylor 10.93 convention)

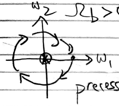
$$\dot{\omega}_2 = \boxed{\frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3} \omega_1 = -\Omega_b \omega_1$$

$$\ddot{\omega}_1 = -\Omega_b^2 \omega_1 \qquad \ddot{\omega}_2 = -\Omega_b^2 \omega_2$$

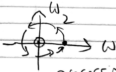
$$\omega_3 \equiv \omega_3 \qquad \omega_1 = \omega_0 \cos \Omega_b t \qquad \omega_2 = -\omega_0 \sin \Omega_b t$$

$$\dot{\omega}_1 = \Omega_b \omega_2 \qquad \dot{\omega}_2 = -\Omega_b \omega_1$$

$\Omega_b > 0$



precession $\sim -\hat{e}_3$



precession $\sim +\hat{e}_3$

The vector representing the precession of $\underset{\sim}{\omega}$ and of $\underset{\sim}{L}$ about $\hat{e}_3$ in the body frame is

$$\underset{\sim}{\Omega}_{body} = -\Omega_b \hat{e}_3 = -\frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3 \hat{e}_3$$

Writing (as we derived last time)

$$\underset{\sim}{\omega} = \frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3$$

(This shows that $\underset{\sim}{\omega}, \underset{\sim}{L}, \hat{e}_3$ are coplanar)

We get

$$\left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} = -\underset{\sim}{\omega} \times \underset{\sim}{L}$$

$$= -\left(\frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3\right) \times \underset{\sim}{L}$$

$$\left(\frac{d\underset{\sim}{L}}{dt}\right)_{body} = \left(-\Omega_b \hat{e}_3\right) \times \underset{\sim}{L}$$

$$= \underset{\sim}{\Omega}_{body} \times \underset{\sim}{L}$$

Since $\underset{\sim}{L}, \underset{\sim}{\omega}, \hat{e}_3$ are coplanar (for $\lambda_1 = \lambda_2$, $\underset{\sim}{\tau} = 0$)

$$\left(\frac{d\underset{\sim}{\omega}}{dt}\right)_{body} = \underset{\sim}{\Omega}_{body} \times \underset{\sim}{\omega}$$

Meanwhile $$\left(\frac{d\hat{e}_3}{dt}\right)_{space} = 0 + \underset{\sim}{\omega} \times \hat{e}_3$$

$$= \left(\frac{\underset{\sim}{L}}{\lambda_1} + \Omega_b \hat{e}_3\right) \times \hat{e}_3$$

$$= \left(\frac{\underset{\sim}{L}}{\lambda_1}\right) \times \hat{e}_3 = \underset{\sim}{\Omega}_{space} \times \hat{e}_3$$

also $$\left(\frac{d\underset{\sim}{\omega}}{dt}\right)_{space} = \underset{\sim}{\Omega}_{space} \times \underset{\sim}{\omega}$$

We derived last time $(\lambda_1 = \lambda_2,\ \underset{\sim}{\tau} = 0)$

$$\frac{\underset{\sim}{L}}{\lambda_1} = \frac{\lambda_1 \omega_1 \hat{e}_1 + \lambda_1 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3}{\lambda_1} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{e}_3$$

$$= \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \underline{\omega_3 \hat{e}_3} + \frac{\lambda_3 \omega_3 \hat{e}_3}{\lambda_1} - \underline{\underline{\omega_3 \hat{e}_3}}$$

$$= \underset{\sim}{\omega} + \frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3 \hat{e}_3 = \underset{\sim}{\omega} + \underset{\sim}{\Omega}_{body}$$

$$= \underset{\sim}{\omega} - \Omega_b \hat{e}_3$$

$$\underset{\sim}{\Omega}_{space} = \frac{\underset{\sim}{L}}{\lambda_1} = \underset{\sim}{\omega} + \underset{\sim}{\Omega}_{body} = \underset{\sim}{\omega} - \Omega_b \hat{e}_3$$

$$\boldsymbol{\omega} \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$$

symmetric top: $\lambda_1 = \lambda_2 \quad \Rightarrow \quad \boldsymbol{L} \;=\; \lambda_1 \omega_1 \hat{\boldsymbol{e}}_1 + \lambda_1 \omega_2 \hat{\boldsymbol{e}}_2 + \lambda_3 \omega_3 \hat{\boldsymbol{e}}_3$

$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3 \;=\; \omega_1 \hat{\boldsymbol{e}}_1 + \omega_2 \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3 + \frac{\lambda_3}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3 - \omega_3 \hat{\boldsymbol{e}}_3$$

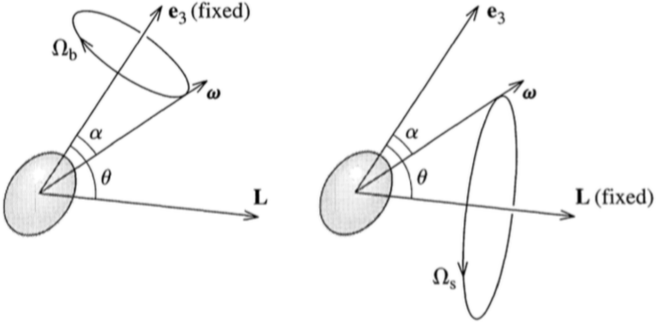
$$\frac{\boldsymbol{L}}{\lambda_1} \;=\; \boldsymbol{\omega} \;+\; \frac{\lambda_3 - \lambda_1}{\lambda_1} \omega_3 \hat{\boldsymbol{e}}_3$$

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \left(\frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \right) \hat{\boldsymbol{e}}_3 \;=\; \frac{\boldsymbol{L}}{\lambda_1} \;+\; \Omega_b \, \hat{\boldsymbol{e}}_3$$

Last line proves that $\boldsymbol{\omega}$, $\boldsymbol{L}$, and $\hat{e}_3$ are coplanar (for $\lambda_1 = \lambda_2$).

Torque-free (10.94): $\boldsymbol{\omega} \;=\; \omega_0 \cos(\Omega_b t) \hat{\boldsymbol{e}}_1 - \omega_0 \sin(\Omega_b t) \hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$

Key trick for understanding “space” and “body” cones: decompose $\boldsymbol{\omega}$ into one part that points along $\boldsymbol{L}$ and one part that points along (or opposite) $\hat{e}_3$. [Sign of Ω_b depends on λ_1 vs. λ_3 magnitudes.]



(a) Body frame (b) Space frame

Figure 10.9 An axially symmetric body (shown here as a prolate spheroid or "egg-shaped" solid) is rotating with angular velocity $\boldsymbol{\omega}$, not in the direction of any of the principal axes. **(a)** As seen in the body frame, both $\boldsymbol{\omega}$ and $\mathbf{L}$ precess about the symmetry axis, $\mathbf{e}_3$, with angular frequency Ω_b given by (10.93). **(b)** As seen in the space frame, $\mathbf{L}$ is fixed, and both $\boldsymbol{\omega}$ and $\mathbf{e}_3$ precess about $\mathbf{L}$ with frequency Ω_s given by (10.96).

Torque-free precession of axially symmetric ($\lambda_1 = \lambda_2$) rigid body

$$\boldsymbol{\omega} \;=\; \frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3 \quad \text{with} \quad \Omega_b \;=\; \frac{\lambda_1 - \lambda_3}{\lambda_1}\,\omega_3$$

$$\boldsymbol{\omega} \;=\; \omega_0 \cos(\Omega_b t)\hat{\boldsymbol{e}}_1 - \omega_0 \sin(\Omega_b t)\hat{\boldsymbol{e}}_2 + \omega_3 \hat{\boldsymbol{e}}_3$$

$\boldsymbol{\Omega}_{\text{space}} = \boldsymbol{L}/\lambda_1$ points along $\boldsymbol{L}$. Describes precession of $\boldsymbol{\omega}$ (and $\hat{\boldsymbol{e}}_3$) about $\boldsymbol{L}$ as seen in space frame.

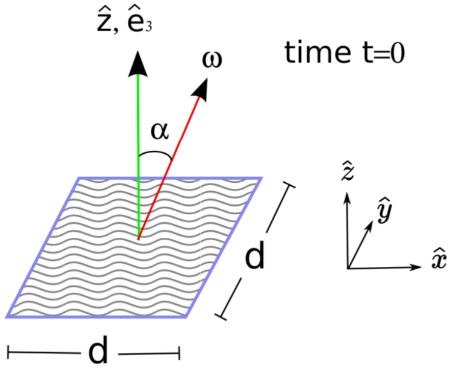
$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_3}{\mathrm{d}t} \;=\; \boldsymbol{\omega}\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\hat{\boldsymbol{e}}_3 \;=\; \left(\frac{\boldsymbol{L}}{\lambda_1}\right)\times\hat{\boldsymbol{e}}_3 \;=\; \boldsymbol{\Omega}_{\text{space}}\times\hat{\boldsymbol{e}}_3$$

$\boxed{\boldsymbol{\Omega}_{\text{body}} = -\Omega_b \hat{\boldsymbol{e}}_3}$ points along $\hat{\boldsymbol{e}}_3$ if $\lambda_3 > \lambda_1$ (oblate, frisbee) and points opposite $\hat{\boldsymbol{e}}_3$ if $\lambda_3 < \lambda_1$ (prolate, US football). Describes precession of $\boldsymbol{\omega}$ (and $\boldsymbol{L}$) about $\hat{\boldsymbol{e}}_3$ as seen in body frame.

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{\text{body}} = -\boldsymbol{\omega}\times\boldsymbol{L} = -\left(\frac{\boldsymbol{L}}{\lambda_1} + \Omega_b \hat{\boldsymbol{e}}_3\right)\times\boldsymbol{L} = (-\Omega_b \hat{\boldsymbol{e}}_3)\times\boldsymbol{L} = \boldsymbol{\Omega}_{\text{body}}\times\boldsymbol{L}$$

$$\boldsymbol{\Omega}_{\text{space}} \;=\; \boldsymbol{\omega} \;+\; \boldsymbol{\Omega}_{\text{body}}$$

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



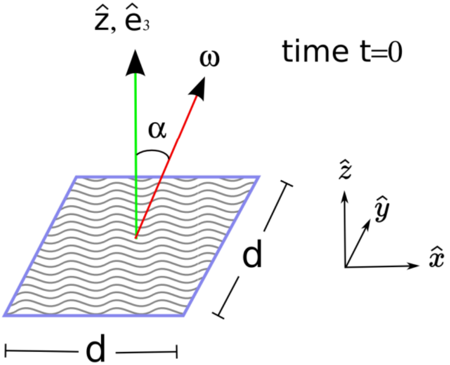
(a) Show that $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0.

(b) Calculate $\boldsymbol{L}$ at $t = 0$.

(c) Sketch $\hat{e}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$.

(d) Draw/label "body cone" and "space cone" on your sketch.

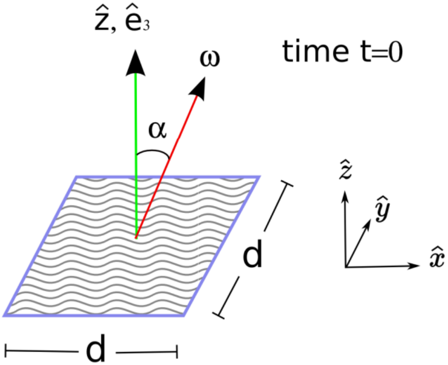
8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\,\hat{\boldsymbol{z}} + \sin\alpha\,\hat{\boldsymbol{x}})$.



(e) Calculate precession frequencies Ω_{body} and Ω_{space}. Indicate directions of precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on drawing.

(f) You argue in HW that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.

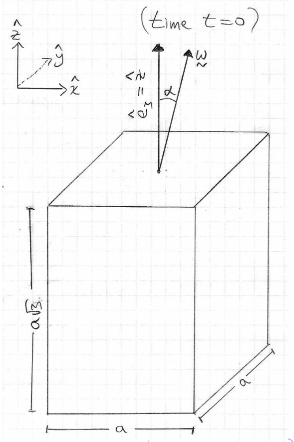


(g) Find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α.

(h) When is this maximum deviation first reached?

video: `https://www.youtube.com/watch?v=oH-dlrIFO1O`

Problem: A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\ a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.



(a) Show (or argue) that the inertia tensor has the form $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and find the constant I_0.

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\boldsymbol{e}}_1$,$\hat{\boldsymbol{e}}_2$,$\hat{\boldsymbol{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

(c) Draw a sketch showing the vectors $\hat{\boldsymbol{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies $\Omega_{\rm body}$ and $\Omega_{\rm space}$. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\bf body}$ and $\boldsymbol{\Omega}_{\bf space}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\bf body}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\bf body}$ when you mean to draw $\boldsymbol{\Omega}_{\bf body}$.

(f) You will argue in a HW problem that $\mathbf{\Omega}_{\mathbf{space}} = \mathbf{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\mathbf{\Omega}_{\mathbf{space}}$ and $\mathbf{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{e}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

(h) At what time t is this maximum deviation first reached?

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

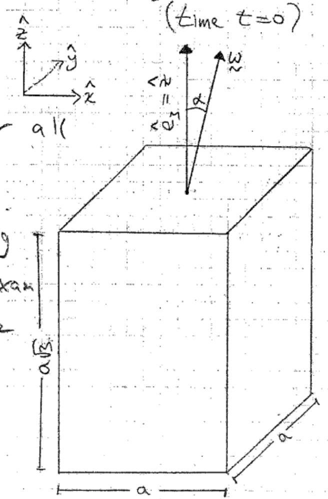
Problem 1.

A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\,a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{z}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{z}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{z} + \sin\alpha\hat{x})$.

(about the CoM)

(a) Show (or argue) that the inertia tensor has the form

$$\underline{\underline{I}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ and find the constant I_0.

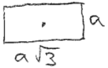


$I_{xy} = -\int xy\,dm = 0$ by symmetry, as for all off-diagonal elements.

$I_{zz} = \int (x^2+y^2)\,dm = \frac{m}{12}(a^2+a^2) = \frac{1}{6}ma^2$ using result for flat plate from back page of exam

$I_{xx} = \int (y^2+z^2)\,dm = \frac{m}{12}\left(a^2+(\sqrt{3}a)^2\right) = \frac{1}{3}ma^2$

using flat plate



So $\boxed{I_0 = \frac{1}{6}ma^2}$

$\lambda_1 = \lambda_2 = 2I_0, \quad \lambda_3 = I_0$

(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{e}_1$,$\hat{e}_2$,$\hat{e}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

$$\underset{\sim}{L} = \underline{\underline{I}}\,\underset{\sim}{\omega} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_3 \omega_3 \hat{e}_3 = 2 I_0 \omega \sin\alpha\, \hat{e}_1 + I_0 \omega \cos\alpha\, \hat{e}_3$$

since at $t=0$ $\quad \hat{e}_1 = \hat{x}$, $\hat{e}_3 = \hat{z}$, $\underset{\sim}{\omega} = \omega \sin\alpha\, \hat{x} + \omega\cos\alpha\, \hat{z}$.

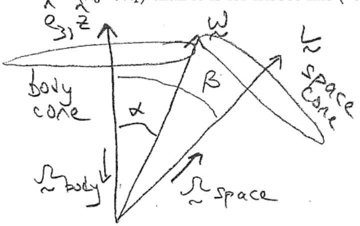
We can also write $\underset{\sim}{L} = 2 I_0 \omega \sin\alpha\, \hat{x} + I_0 \omega \cos\alpha\, \hat{z}$.

This second expression remains true for $t>0$, because

$$0 = \underset{\sim}{\tau} = \left(\frac{d\underset{\sim}{L}}{dt}\right)_{\substack{\text{space}\\ \text{frame}}} \Rightarrow \underset{\sim}{L} \text{ is constant in space frame for torque-free rotation.}$$

The unit vectors $\hat{e}_i$ will rotate with the body. In particular, we will see L_1 and L_2 are not constant.

(c) Draw a sketch showing the vectors $\hat{\mathbf{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).



$$\tan\alpha = \frac{\omega_1}{\omega_3} \qquad \tan\beta = \frac{L_1}{L_3} = 2\tan\alpha$$

body cone is traced out by ω as it precesses about $\hat{e}_3$ in body frame.

space cone is traced out by ω as it precesses about L in space frame.

Note that L, ω, $\hat{e}_3$ remain coplanar:

$$\left.\begin{aligned}
\hat{e}_3 &= 0\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \hat{e}_3 \\
L &= \lambda_1\cdot(\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \lambda_3\omega_3\hat{e}_3 \\
\omega &= (\omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2) + \omega_3\hat{e}_3
\end{aligned}\right\}$$

coplanar: plane defined by $\hat{e}_3$ and $\omega_1\hat{e}_1 + \omega_2\hat{e}_2 = \vec{\omega}_\perp$

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\text{body}}$ and $\boldsymbol{\Omega}_{\text{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\text{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\text{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\text{body}}$.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\text{body}}$ and $\boldsymbol{\Omega}_{\text{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\text{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\text{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\text{body}}$.

$$\underset{\sim}{\Omega}_{space} = \frac{\underset{\sim}{L}}{\lambda_1} = \omega \sin\alpha\,\hat{x} + \frac{\lambda_3}{\lambda_1}\omega\cos\alpha\,\hat{z} = \omega\left(\sin\alpha\,\hat{x} + \tfrac{1}{2}\cos\alpha\,\hat{z}\right)$$

$$= \left(\frac{\omega}{2}\sqrt{\cos^2\alpha + 4\sin^2\alpha}\right)\hat{L} = \left(\frac{\omega}{2}\sqrt{1+3\sin^2\alpha}\right)\hat{L}$$

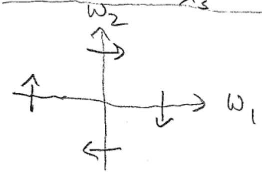
At $t=0$, $\underset{\sim}{\Omega}_{space} = \omega\sin\alpha\,\hat{e}_1 + \tfrac{1}{2}\omega\cos\alpha\,\hat{e}_3$

$$\underset{\sim}{\Omega}_{body} = \frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\hat{e}_3 = \frac{I_0 - 2I_0}{2I_0}\omega_3\hat{e}_3 = -\tfrac{1}{2}\omega\cos\alpha\,\hat{e}_3$$

$$\boxed{\dot{\omega}_3 = \omega_1\omega_2\frac{\lambda_1-\lambda_2}{\lambda_3} = 0}$$

$$\dot{\omega}_1 = \omega_2\omega_3\frac{\lambda_2-\lambda_3}{\lambda_1} = \tfrac{1}{2}\omega_3\omega_2 = \left(\tfrac{1}{2}\omega\cos\alpha\right)\omega_2$$

$$\dot{\omega}_2 = \omega_3\omega_1\frac{\lambda_3-\lambda_1}{\lambda_2} = -\tfrac{1}{2}\omega_3\omega_1 = -\left(\tfrac{1}{2}\omega\cos\alpha\right)\omega_1$$



clockwise $\rightarrow$ $\underset{\sim}{\omega}$ precesses in $-\hat{e}_3$ direction in body frame

(f) You argued in HW11 that $\Omega_{\textbf{space}} = \Omega_{\textbf{body}} + \omega$. Verify (by writing out components) that this relationship holds for the $\Omega_{\textbf{space}}$ and $\Omega_{\textbf{body}}$ that you calculate for $t = 0$.

At $t=0$, $\underset{\sim}{\Omega}_{body} = -\frac{1}{2}\omega\cos\alpha\,\hat{e}_3$

$\underset{\sim}{\omega} = \omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

add to $+\frac{1}{2}\omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha\,\hat{e}_1$

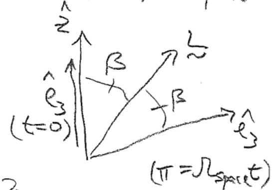
At $t=0$, $\underset{\sim}{\Omega}_{space} = \omega\sin\alpha\,\hat{e}_1 + \frac{1}{2}\omega\cos\alpha\,\hat{e}_3$ ✓

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{z}$ and $\hat{e}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

The initial angle between $\hat{e}_3$ and $\underset{\sim}{L}$ is $\beta = \text{atan}(2\tan\alpha) \simeq 2\alpha$.

This angle is a constant of the motion, because $L_3 = \lambda_3 \omega_3 = \text{const.}$ and $\underset{\sim}{L}_{\text{space}}$ (hence magnitude of $\underset{\sim}{L}_{\text{body}}$) is constant. As shown on diagram (c), $\underset{\sim}{L}, \underset{\sim}{\omega}, \hat{e}_3$ remain coplanar. So in the space frame, $\underset{\sim}{\omega}$ and $\hat{e}_3$ both precess about $\underset{\sim}{L}$ with frequency Ω_{space}.

Maximum angle between $\hat{e}_3$ and $\hat{z}$ is $\boxed{2\beta \simeq 4\alpha}$.



(h) At what time t is this maximum deviation first reached?

one-half period of Ω_{space}:

$$\Omega_{\text{space}} t = \pi$$

$$t = \frac{\pi}{\Omega_{\text{space}}} = \frac{2\pi}{\omega\sqrt{1+3\sin^2\alpha}} \simeq \frac{2\pi}{\omega} \quad \left(\simeq \frac{2\pi}{\omega_3} \text{ if } \alpha \ll 1\right)$$

So the precession ("wobble") has $\simeq$ half the frequency of the "spin" if $\alpha \ll 1$.

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

We won't go over this, as Prof Liu covered this topic last week, but I'll leave it here as an example. You might especially find it useful that Mathematica (or Wolfram Alpha) can easily find eigenvalues and eigenvectors for you. (See later pages.)

(Taylor 10.35) A rigid body consists of:

m at $(a, 0, 0) \ = \ a(1, \ 0, \ 0)$

$2m$ at $(0, a, a) \ = \ a(0, \ 1, \ 1)$

$3m$ at $(0, a, -a) \ = \ a(0, \ 1, -1)$

Find inertia tensor $\underline{\underline{\boldsymbol{I}}}$, its principal moments, and the principal axes.

$$I_{xx} = \sum m(y^2+z^2) = ma^2(2\cdot 2+2\cdot 3) = 10ma^2$$

$$I_{yy} = \sum m(x^2+z^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{zz} = \sum m(x^2+y^2) = ma^2(1+2+3) = 6ma^2$$

$$I_{xy} = -\sum mxy = -ma^2(0) = 0$$

$$I_{xz} = -\sum mxz = -ma^2(0) = 0$$

$$I_{yz} = -\sum myz = -ma^2(2-3) = ma^2$$

$$\underline{\underline{I}} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} ma^2$$

$$\underline{\underline{I}}\ \underset{\sim}{\omega} = \lambda\,\underset{\sim}{\omega} \Rightarrow \left(\underline{\underline{I}} - \lambda\,\mathbb{1}\right)\underset{\sim}{\omega} = 0$$

$$\Rightarrow \det\left(\underline{\underline{I}} - \lambda\,\mathbb{1}\right) = 0$$

$$0 = (10-\lambda)(6-\lambda)^2 - (10-\lambda) \Rightarrow \lambda = 10 \text{ or } (6-\lambda)^2 = 1$$

$$6-\lambda = 1 \Rightarrow \lambda = 5, \quad 6-\lambda = -1 \Rightarrow \lambda = 7 \qquad \lambda \in \{10, 7, 5\}$$

$$\lambda \in \{10, 7, 5\}\, ma^2$$

$$0 = \begin{pmatrix} 10-10 & 0 & 0 \\ 0 & 6-10 & 1 \\ 0 & 1 & 6-10 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4y+z \\ y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = z_1 = 0 \Rightarrow \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-7 & 0 & 0 \\ 0 & 6-7 & 1 \\ 0 & 1 & 6-7 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3x \\ z-y \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0,\ y_2 = z_2 \Rightarrow \hat{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 10-5 & 0 & 0 \\ 0 & 6-5 & 1 \\ 0 & 1 & 6-5 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 5x \\ y+z \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3 = 0,\ y_3 = -z_3 \Rightarrow \hat{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

eigenvectors {{10,0,0},{0,6,1},{0,1,6}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}\right]$$

Results:

$$v_1 = (1, 0, 0)$$

$$v_2 = (0, 1, 1)$$

$$v_3 = (0, -1, 1)$$

Corresponding eigenvalues:

$$\lambda_1 = 10$$

$$\lambda_2 = 7$$

$$\lambda_3 = 5$$

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In[1]:= m = {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

Out[1]= {{10, 0, 0}, {0, 6, 1}, {0, 1, 6}}

In[2]:= MatrixForm[m]

2]//MatrixForm=
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$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

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In[3]:= Eigenvalues[m]

Out[3]= {10, 7, 5}

In[4]:= Eigenvectors[m]

Out[4]= {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}

In[5]:= Eigensystem[m]

Out[5]= {{10, 7, 5},
  {{1, 0, 0}, {0, 1, 1}, {0, -1, 1}}}
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