You know that representing the orientation in space of a rigid body requires us to specify 3 numbers — 3 rotational degrees of freedom.

One way to do this is to write out the (x,y,z) coordinates of $\hat{e}_1$, of $\hat{e}_2$, and of $\hat{e}_3$. But it turns out, as we'll see, that if you're given the three EULER ANGLES, ϕ, θ, ψ, then you can use these 3 angles to compute the (x,y,z) coordinates of the 3 body axes.

So the Euler angles ~~give you the spatial coordinates~~ map the body axes 1,2,3 into x,y,z space.

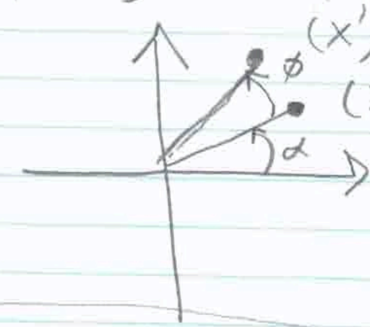
Suppose we start with a vector

$$(x,y) = (R\cos\alpha, R\sin\alpha)$$

and we want to rotate it (about the origin) by an angle ϕ. How do we write the new coordinates (x', y') as linear combinations of the old coordinates (x,y)?

$$(x', y') = R(\cos(\alpha+\phi), \sin(\alpha+\phi))$$

$$(x,y) = (R\cos\alpha, R\sin\alpha)$$



$$x' = R\cos\alpha\cos\phi - R\sin\alpha\sin\phi$$
$$x' = x\cos\phi - y\sin\phi$$

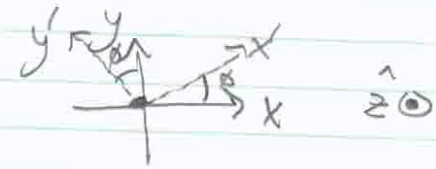
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$y' = R\sin\alpha\cos\phi + R\cos\alpha\sin\phi$$
$$y' = y\cos\phi + x\sin\phi$$

chat:

writing that as a 3x3 matrix, for 3D space,

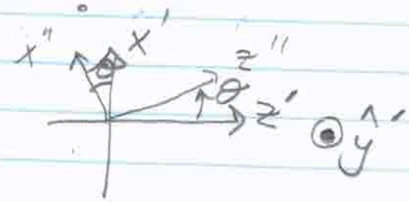
- rotate by angle ϕ about $\hat{z}$:



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x' = x\cos\phi - y\sin\phi \qquad y' = x\sin\phi + y\cos\phi \qquad z' = z$$

- Next, rotate by angle θ about $\hat{y}'$:



$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$x'' = x'\cos\theta + z'\sin\theta \qquad y'' = y' \qquad z'' = -x'\sin\theta + z'\cos\theta$$

check ✓

- Mnemonic: for infinitesimal rotation angle $\varepsilon \ll 1$,

$$\vec{r} \to \vec{r} + \varepsilon\, \hat{\omega} \times \vec{r}$$

So for rotation about $\hat{y}$

$$\vec{r} \to \vec{r} + \varepsilon\, \hat{y} \times \vec{r}$$

$$(1,0,0) \to (1,0,0) + \varepsilon\,(0,1,0) \times (1,0,0) = (1,0,-\varepsilon)$$

$$\hat{x} \to \hat{x} + \varepsilon\, \hat{y} \times \hat{x} = \hat{x} - \varepsilon\hat{z}$$

$$\begin{aligned}(x'', y'', z'') &= (x', y', z') + \varepsilon\, \hat{y} \times (x', y', z') \\ &= (x', y', z') + \varepsilon\, \hat{y} \times (x'\hat{x} + y'\hat{y} + z'\hat{z}) \\ &= (x', y', z') + \varepsilon\,(-x'\hat{z} + z'\hat{x}) \\ &= (x' + \varepsilon z',\ y',\ z' - \varepsilon x')\end{aligned}$$

The hardest part of writing down 3x3 rotation matrices is remembering where to put the minus sign.

$$\begin{matrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{matrix}$$

Once you've worked out one case correctly (e.g. from a diagram), here's a trick (thanks to 2015 student Adam Zachar) for working out the other two.

about $\hat{z}$

about $\hat{y}$

$$\begin{matrix} \cos & -\sin & 0 & \cos & -\sin \\ \sin & \cos & 0 & \sin & \cos \\ 0 & 0 & 1 & 0 & 0 \\ \cos & -\sin & 0 & \cos & -\sin \\ \sin & \cos & 0 & \sin & \cos \end{matrix}$$

about $\hat{x}$

You can double-check these results using Mathematica:

~~RotationMatrix[φ, {0,0,1}] // MatrixForm~~

```
MatrixForm[RotationMatrix[φ, {0,0,1}]]
MatrixForm[RotationMatrix[θ, {0,1,0}]]
MatrixForm[RotationMatrix[α, {1,0,0}]]
```

(This 3rd one is <u>not</u> an Euler angle)

Euler angles: can move (x, y, z) axes to arbitrary orientation

Rotate by ϕ about $\hat{z}$

Then rotate by θ about $\hat{y}'$ $(\hat{e}_2')$

Then rotate by ψ about $\hat{z}''$ $(\hat{e}_3)$

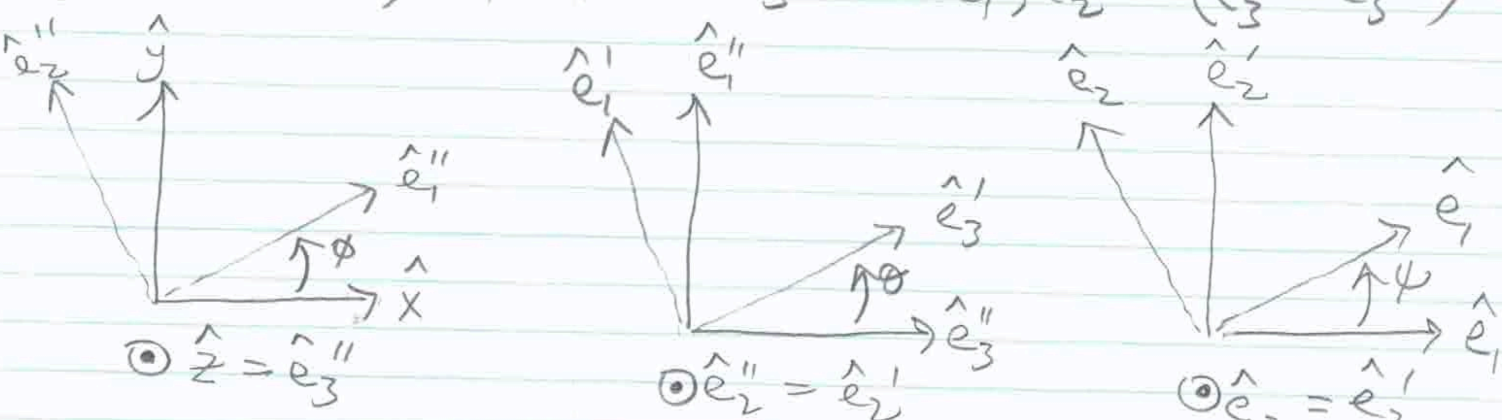
$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} c\theta c\phi c\psi - s\phi s\psi & -c\theta s\phi c\psi - c\phi s\psi & s\theta c\psi \\ c\theta c\phi s\psi + s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\psi \\ -s\theta c\phi & s\theta s\phi & c\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

```
RotationMatrix[φ] // MatrixForm
RotationMatrix[φ, {0, 0, 1}] // MatrixForm
RotationMatrix[θ, {0, 1, 0}] / MatrixForm
r1 = RotationMatrix[φ, {0, 0, 1}];
r2 = RotationMatrix[θ, {0, 1, 0}];
r3 = RotationMatrix[ψ, {0, 0, 1}];
r3 . r2 . r1 // MatrixForm
```

- Let the Euler angles ϕ, θ, ψ vary with time, as the body rotates.
- I will write out more steps than Taylor does, and I may confuse you by saying $(\hat{x}, \hat{y}, \hat{z}) \to (\hat{e}_1'', \hat{e}_2'', \hat{e}_3'') \to (\hat{e}_1', \hat{e}_2', \hat{e}_3') \to (\hat{e}_1, \hat{e}_2, \hat{e}_3)$
- I do this so that my $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ are the same as Taylor's.

1. Rotate by ϕ about $\hat{z} \to \hat{e}_1'', \hat{e}_2''$ $\quad (\hat{e}_3'' = \hat{z})$
2. Rotate by θ about $\hat{e}_2'' \to \hat{e}_1', \hat{e}_3'$ $\quad (\hat{e}_2' = \hat{e}_2'')$
3. Rotate by ψ about $\hat{e}_3' \to \hat{e}_1, \hat{e}_2$ $\quad (\hat{e}_3 = \hat{e}_3')$



$$\vec{\omega} = \dot{\phi}\,\hat{z} + \dot{\theta}\,\hat{e}_2'' + \dot{\psi}\,\hat{e}_3'$$
$$\vec{\omega} = \dot{\phi}\,\hat{z} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$$

Remarkable trick: we can write $\vec{\omega}$ as the vector sum of 3 separate angular-velocity vectors, about 3 successive axes.

Next: project $\vec{\omega}$ onto more convenient sets of unit vectors.

Looking at the 3 diagrams, we can read off orthogonal matrix:

$$\hat{e}_1'' = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{x} = \hat{e}_1''\cos\phi - \hat{e}_2''\sin\phi$$
$$\hat{e}_2'' = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{y} = \hat{e}_1''\sin\phi + \hat{e}_2''\cos\phi$$
$$\hat{e}_3'' = \hat{z} \qquad \hat{z} = \hat{e}_3''$$

(inverse = transpose)

$$\hat{e}_1' = \hat{e}_1''\cos\theta - \hat{e}_3''\sin\theta \qquad \hat{e}_1'' = \hat{e}_1'\cos\theta + \hat{e}_3'\sin\theta$$
$$\hat{e}_3' = \hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta \qquad \hat{e}_3'' = -\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta$$
$$\hat{e}_2' = \hat{e}_2'' \qquad \hat{e}_2'' = \hat{e}_2'$$

$$\hat{e}_1 = \hat{e}_1'\cos\psi + \hat{e}_2'\sin\psi \qquad \hat{e}_1' = \hat{e}_1\cos\psi - \hat{e}_2\sin\psi$$
$$\hat{e}_2 = -\hat{e}_1'\sin\psi + \hat{e}_2'\cos\psi \qquad \hat{e}_2' = \hat{e}_1\sin\psi + \hat{e}_2\cos\psi$$
$$\hat{e}_3 = \hat{e}_3' \qquad \hat{e}_3' = \hat{e}_3$$

Let's first find $\vec{\omega}$ in space axes:

$$\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3'$$
$$= \dot{\phi}\hat{z} + \dot{\theta}(-\hat{x}\sin\phi + \hat{y}\cos\phi) + \dot{\psi}(\hat{e}_1''\sin\theta + \hat{e}_3''\cos\theta)$$
$$= \dot{\phi}\hat{z} - \dot{\theta}\hat{x}\sin\phi + \dot{\theta}\hat{y}\cos\phi + \dot{\psi}\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \dot{\psi}\cos\theta(\hat{z})$$

$$\omega_x = -\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi \qquad \omega_y = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \qquad \omega_z = \dot{\phi} + \dot{\psi}\cos\theta$$

$$\vec{\omega}_{\text{space axes}} = \left(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi,\ \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,\ \dot{\phi} + \dot{\psi}\cos\theta\right)$$

Then let's find $\vec{\omega}$ in the body axes:

$$\vec{\omega} = \dot{\phi}\hat{e}_3'' + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3 = \dot{\phi}(-\hat{e}_1'\sin\theta + \hat{e}_3'\cos\theta) + \dot{\theta}(\hat{e}_1\sin\psi + \hat{e}_2\cos\psi) + \dot{\psi}\hat{e}_3$$

$$\vec{\omega} = -\dot{\phi}\sin\theta(\hat{e}_1\cos\psi - \hat{e}_2\sin\psi) + \dot{\phi}\cos\theta(\hat{e}_3) + \dot{\theta}\sin\psi\,\hat{e}_1 + \dot{\theta}\cos\psi\,\hat{e}_2 + \dot{\psi}\hat{e}_3$$

$$\omega_1 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi \qquad \omega_2 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \qquad \omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

$$\vec{\omega}_{\text{body axes}} = (\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi,\ \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi,\ \dot{\psi} + \dot{\phi}\cos\theta)$$

For the symmetric top ($\lambda_1 = \lambda_2$), it will be most convenient to express $\vec{\omega}$ in the "primed" basis, i.e. before the final rotation about $\hat{e}_3$. This is the same as ω_{body} when $\psi = 0$:

$$\vec{\omega} = (-\dot{\phi}\sin\theta)\hat{e}_1' + (\dot{\theta})\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3'$$

The "primed" basis is most convenient for writing $\vec{L}$, T, $\mathcal{L}$

$$\vec{L} = (-\lambda_1\dot{\phi}\sin\theta)\hat{e}_1' + (\lambda_1\dot{\theta})\hat{e}_2' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3'$$

$$T = \tfrac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$U = MgR\cos\theta$$

$$\mathcal{L} = \tfrac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

2 ignorable coordinates: ϕ and ψ $\Rightarrow$ can reduce θ EOM to 1-variable problem

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)} \equiv P_\psi \equiv \text{constant} = \lambda_3\omega_3 = \boxed{L_3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + \boxed{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}\cos\theta \equiv P_\phi \equiv \text{constant}$$

$$= \boxed{L_z - L_3\cos\theta} + \boxed{L_3}\cos\theta = L_z$$

So $\boxed{\begin{matrix} P_\psi \equiv L_3 \\ P_\phi \equiv L_z \end{matrix}}$

Note: $L_z = \vec{L}\cdot\hat{z} = \vec{L}\cdot(\hat{e}_3'') = -L_1'\sin\theta + L_3'\cos\theta = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$

$$L_z = \boxed{\lambda_1\dot{\phi}\sin^2\theta} + L_3\cos\theta$$

$$\Rightarrow \boxed{\lambda_1\dot{\phi}\sin^2\theta} = \boxed{L_z - L_3\cos\theta}$$

Find the θ EOM:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) \Rightarrow \lambda_1 \dot\phi^2 \sin\theta\cos\theta + \lambda_3(\dot\psi + \dot\phi\cos\theta)(-\dot\phi\sin\theta) + MgR\sin\theta =$$

$$= \frac{d}{dt}\left(\lambda_1 \dot\theta\right) = \lambda_1 \ddot\theta$$

$$\lambda_1 \ddot\theta = \lambda_1 \dot\phi^2 \sin\theta\cos\theta - \lambda_3 \boxed{(\dot\psi + \dot\phi\cos\theta)} \dot\phi\sin\theta + MgR\sin\theta$$

$$\boxed{\lambda_1 \ddot\theta = \lambda_1 \dot\phi^2 \sin\theta\cos\theta - \lambda_3 \boxed{\omega_3} \dot\phi \sin\theta + MgR\sin\theta}$$ θ EOM

First consider the case where $\theta = \text{constant}$ (precession without nutation)

recall $\lambda_1 \dot\phi \sin^2\theta = L_z - L_3\cos\theta$. If $\theta = \text{const.}$ then $\dot\phi = \text{const} \equiv \Omega$

$$\dot\phi = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta} = \text{constant if } \theta \text{ is constant.}$$

$$0 = \lambda_1 \Omega^2 \cancel{\sin\theta} \cos\theta - \lambda_3\omega_3 \Omega \cancel{\sin\theta} + MgR\cancel{\sin\theta}$$

$$(\lambda_1\cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR = 0$$

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1\cos\theta MgR}}{2\lambda_1\cos\theta}$$

$$\Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4\lambda_1\cos\theta MgR}{(\lambda_3\omega_3)^2}}\right)$$

has 2 real solutions if $(\lambda_3\omega_3)^2 > 4\lambda_1 MgR\cos\theta$
("if ω_3 is large enough")

math is simplest if $(\lambda_3\omega_3)^2 \gg 4\lambda_1 MgR\cos\theta$ ("ω_3 is very large")

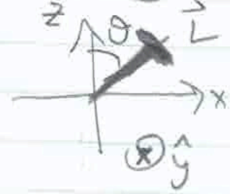
$$\text{Then } \Omega = \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \left(1 - \frac{2\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2}\right)\right)$$

$$\Omega_+ \simeq \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} = \text{familiar } \Omega_{space} \text{ for zero-torque precession}$$

$$\Omega_- \simeq \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\cdot\frac{2\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2} = \boxed{\frac{MgR}{\lambda_3\omega_3} \simeq \Omega_-}$$

This is same result as $\vec{\Gamma} = \frac{d\vec{L}}{dt} \Rightarrow MgR\sin\theta = \lambda_3\omega_3\Omega\sin\theta$

"freshman physics" for precessing top



$$(L\sin\theta)\Omega\hat{y} = \frac{d\vec{L}}{dt} = (R)(Mg)\sin\theta\ \hat{y}$$

$$\Omega = \frac{MgR}{L} = \frac{MgR}{\lambda_3\omega_3} \quad ✓$$

Now look at 1-variable problem for non-constant θ:

$$E = T + U = \frac{1}{2}\lambda_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3\underbrace{(\dot{\psi}+\dot{\phi}\cos\theta)^2}_{\omega_3^2} + MgR\cos\theta$$

$$\lambda_1\dot{\phi}\sin^2\theta = L_z - L_3\cos\theta$$

$$\Rightarrow \dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1\sin^2\theta}$$

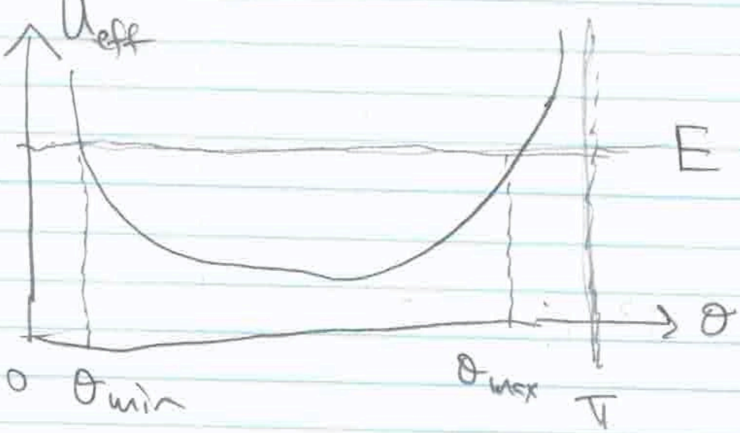
$$E = \frac{1}{2}\lambda_1\frac{(L_z - L_3\cos\theta)^2\cancel{\sin^2\theta}}{\lambda_1^2\sin^2\theta\cancel{\sin^2\theta}} + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3\omega_3^2 + MgR\cos\theta$$

$$E = \frac{\lambda_1\dot{\theta}^2}{2} + \underbrace{\boxed{\frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$$

(one-dimensional problem)

θ "nutates" back & forth between θ_{min} and θ_{max}



YouTube videos (see very last slide)