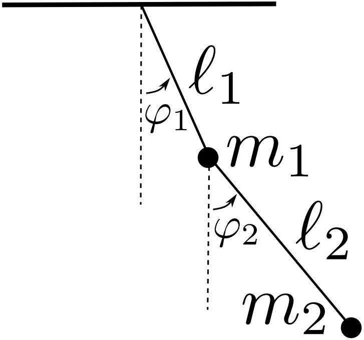
Physics 3351 — Wednesday, April 5, 2023

- I'm Bill Ashmanskas (Bill, Dr Bill, Prof Bill, etc), filling in for Prof Sako today. I taught Phys 351 in 2015, 2017, 2018.
- Our task today is basically to finish Taylor ch11 (coupled oscillators), so that Prof Sako can start some fun new material with you on Monday, and so that in the meantime, you're well prepared to work some ch11 HW problems.
- You should **READ** §11.5 ("the general case") on your own, as it is worth seeing, but I don't think our going through it together on the board will add any value.
- We will spend today's class working two example problems together: the double pendulum (which some of you already worked as an XC problem), and three coupled oscillators.
- These slides will be on Canvas/files/notes as well as at

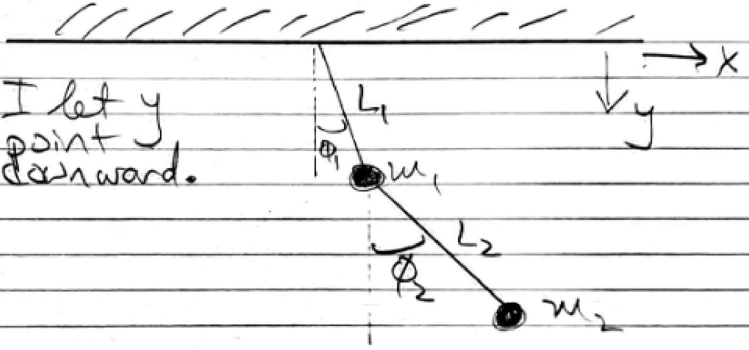
`http://positron.hep.upenn.edu/p351/files/p351_notes_20230405.pdf`

- Instead of writing down what you see me write on the board, I want you instead to work, preferably with a friend, through your parallel calculations to mine. I'll give you a head start.

Consider a double pendulum consisting of two bobs confined to move in a plane. The rods are of length ℓ_1 and ℓ_2, respectively, and the bobs are of mass m_1 and m_2, respectively. The generalized coordinates used to describe the system are φ_1 and φ_2, the angles that the rods make with the vertical.



Let's write the potential energy U in terms of m_1, m_2, ℓ_1, ℓ_2, and the gravitational acceleration g at Earth's surface. **You go first!**



$$y_1 = L_1 \cos\phi_1 \qquad y_2 = y_1 + L_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$ (with my choice of y-axis)

$$U = -m_1 g L_1 \cos\phi_1 - m_2 g (L_1 \cos\phi_1 + L_2 \cos\phi_2)$$

OK, next write the kinetic energy. You might want to try it two different ways. You can save yourself a bunch of tedious algebra in writing the KE of m_2 by using the trick that $\vec{v}_2 = \vec{v}_1 + \vec{v}_{21}$ where $\vec{v}_{21} = \vec{v}_2 - \vec{v}_1$.

Then notice that

$$|\vec{v}_2|^2 = |\vec{v}_1|^2 + |\vec{v}_{21}|^2 + 2|\vec{v}_1|\,|\vec{v}_{21}|\,\cos(\Delta\phi)$$

where $\Delta\phi$ is the angle between $\vec{v}_1$ and $\vec{v}_{21}$.

As a check, if you have time, also try using the brute-force method of writing down x_1, y_1, x_2, y_2 and differentiating. (Yuck.)

$$\underset{\sim}{r}_2 = \underset{\sim}{r}_1 + \underset{\sim}{r}_{21}$$

$$\dot{\underset{\sim}{r}}_2 = \dot{\underset{\sim}{r}}_1 + \dot{\underset{\sim}{r}}_{21}$$

$$(\dot{\underset{\sim}{r}}_2)^2 = (\dot{\underset{\sim}{r}}_1)^2 + (\dot{\underset{\sim}{r}}_{21})^2 + 2\dot{\underset{\sim}{r}}_1 \cdot \dot{\underset{\sim}{r}}_{21}$$

$$= \dot{r}_1^2 + \dot{r}_{21}^2 + 2\dot{r}_1\dot{r}_{21} \cos(\measuredangle \text{ betw } \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_{21})$$

$$(\dot{\underset{\sim}{r}}_1)^2 = (L_1\dot{\phi}_1)^2 \qquad (\dot{\underset{\sim}{r}}_{21})^2 = (L_2\dot{\phi}_2)^2$$

$\measuredangle$ between $\dot{\underset{\sim}{r}}_1$, $\dot{\underset{\sim}{r}}_{21}$ is just $\phi_2 - \phi_1$

So $2\dot{\underset{\sim}{r}}_1 \cdot \dot{\underset{\sim}{r}}_{21} = 2(L_1\dot{\phi}_1)(L_2\dot{\phi}_2)\cos(\phi_2 - \phi_1)$

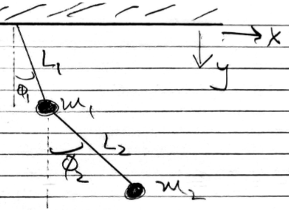
So $|\dot{\underset{\sim}{r}}_2|^2 = L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_2 - \phi_1)$

$$T = \frac{1}{2}m_1|\dot{\underset{\sim}{r}}_1|^2 + \frac{1}{2}m_2|\dot{\underset{\sim}{r}}_2|^2 =$$

$$\frac{1}{2}m_1L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2\left[L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_2 - \phi_1)\right]$$

Next slide: brute-force method for KE. I had Mathematica do the tedious algebra! License is free for all SAS students.

I let y point downward.



$y_1 = L_1 \cos\phi_1 \qquad y_2 = y_1 + L_2 \cos\phi_2$

$x_1 = L_1 \sin\phi_1 \qquad x_2 = x_1 + L_2 \sin\phi_2$

```
ClearAll["Global`*"];
```

```
y1 := L1 Cos[ϕ1[t]] ;
x1 := L1 Sin[ϕ1[t]] ;
y2 := y1 + L2 Cos[ϕ2[t]] ;
x2 := x1 + L2 Sin[ϕ2[t]];
x1dot := D[x1, t];
y1dot := D[y1, t];
x2dot := D[x2, t];
y2dot := D[y2, t];
T := (m1 / 2) (x1dot^2 + y1dot^2) +
   (m2 / 2) (x2dot^2 + y2dot^2)
FullSimplify[T]
```

$$\frac{1}{2}\left(L1^2\,(m1 + m2)\,\phi1'[t]^2 + 2\,L1\,L2\,m2\,\mathrm{Cos}[\phi1[t] - \phi2[t]]\,\phi1'[t]\,\phi2'[t] + L2^2\,m2\,\phi2'[t]^2\right)$$

Either way, we get

$$\mathcal{L} \;=\; T - U \;=$$

$$\frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 \;+\; \frac{1}{2}m_2 L_2^2\dot{\phi}_2^2 \;+\; m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$+\; (m1 + m2) g L_1 \cos\phi_1 \;+\; m_2 g L_2 \cos\phi_2$$

```
T := (m1 / 2) (x1dot ^ 2 + y1dot ^ 2) +
  (m2 / 2) (x2dot ^ 2 + y2dot ^ 2)
FullSimplify[T]
```

$$\frac{1}{2}\left(L1^2\,(m1 + m2)\,\phi 1'[t]^2 + 2\,L1\,L2\,m2\,\mathrm{Cos}[\phi 1[t] - \phi 2[t]]\,\phi 1'[t]\,\phi 2'[t] + L2^2\,m2\,\phi 2'[t]^2\right)$$

```
(* note that I defined y axis to point downward *)
U := -m1 g y1 - m2 g y2
L := T - U
FullSimplify[L]
```

$$\frac{1}{2}\left(2\,g\,(L1\,(m1 + m2)\,\mathrm{Cos}[\phi 1[t]] + L2\,m2\,\mathrm{Cos}[\phi 2[t]]) + L1^2\,(m1 + m2)\,\phi 1'[t]^2 + 2\,L1\,L2\,m2\,\mathrm{Cos}[\phi 1[t] - \phi 2[t]]\,\phi 1'[t]\,\phi 2'[t] + L2^2\,m2\,\phi 2'[t]^2\right)$$

If we take this Lagrangian at face value, we'll get a pair of coupled nonlinear ODEs. At large amplitude, the double pendulum's motion in fact exhibits chaotic behavior (Taylor ch12).

```
L := T - U
FullSimplify[L]
```

$$\frac{1}{2}\Big(2\,g\,(L1\,(m1+m2)\,\text{Cos}[\phi1[t]] + L2\,m2\,\text{Cos}[\phi2[t]]) + L1^2\,(m1+m2)\,\phi1'[t]^2 + 2\,L1\,L2\,m2\,\text{Cos}[\phi1[t]-\phi2[t]]\,\phi1'[t]\,\phi2'[t] + L2^2\,m2\,\phi2'[t]^2\Big)$$

```
eomexpr1 := D[D[L, ϕ1'[t]], t] == D[L, ϕ1[t]]
eom1 := FullSimplify[eomexpr1]
eom1
```

$$L1\,\Big(L2\,m2\,\text{Sin}[\phi1[t]-\phi2[t]]\,\phi2'[t]^2 + (m1+m2)\,(g\,\text{Sin}[\phi1[t]] + L1\,\phi1''[t]) + L2\,m2\,\text{Cos}[\phi1[t]-\phi2[t]]\,\phi2''[t]\Big) = 0$$

```
eomexpr2 := D[D[L, ϕ2'[t]], t] == D[L, ϕ2[t]]
eom2 := FullSimplify[eomexpr2]
eom2
```

$$L2\,m2\,\Big(g\,\text{Sin}[\phi2[t]] - L1\,\text{Sin}[\phi1[t]-\phi2[t]]\,\phi1'[t]^2 + L1\,\text{Cos}[\phi1[t]-\phi2[t]]\,\phi1''[t] + L2\,\phi2''[t]\Big) = 0$$

Mathematica has no trouble numerically integrating these two coupled ODEs (as I'll show later), but we seek a less messy solution.

We prefer to study the relatively simple and elegant motion that results at small amplitude. To do this, we have two options:

- Simplify $\mathcal{L}$ to 2nd order in ϕ_1, ϕ_2, $\dot{\phi}_1$, $\dot{\phi}_2$. Or else,
- get full EOMs and simplify them to 1st order in ϕ_1, ϕ_2, $\dot{\phi}_1$, $\dot{\phi}_2$.

The first option is less effort, as we drop unwanted terms sooner.

$$\mathcal{L} \;=\; T - U \;=$$

$$\frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 \;+\; \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 \;+\; m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$$
$$+\;(m1+m2)gL_1\cos\phi_1 \;+\; m_2gL_2\cos\phi_2$$

Let's reduce $\mathcal{L}$ to $\boxed{\text{2nd order}}$ in ϕ, $\dot{\phi}$ and their products. Then use your simplified $\mathcal{L}$ to find the two EOMs using the Euler-Lagrange equation. **You try, then I'll follow.**

$\cos(\phi_1-\phi_2) \to 1$ b/c it already multiplies $\dot{\phi}_1\dot{\phi}_2$

$\cos\phi_1 \to (1-\frac{1}{2}\phi_1)$ $\qquad$ $\cos\phi_2 \to (1-\frac{1}{2}\phi_2)$

$$\mathcal{L} \to \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2$$
$$+(m_1+m_2)gL_1\left(1-\tfrac{1}{2}\phi_1^2\right) + m_2gL_2\left(1-\tfrac{1}{2}\phi_2^2\right)$$

$$\frac{\partial\mathcal{L}}{\partial\phi_1} = -(m_1+m_2)gL_1\phi_1 \qquad \frac{\partial\mathcal{L}}{\partial\phi_2} = -m_2gL_2\phi_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = (m_1+m_2)L_1^2\dot{\phi}_1 + m_2L_1L_2\dot{\phi}_2$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}\right) = (m_1+m_2)L_1^2\ddot{\phi}_1 + m_2L_1L_2\ddot{\phi}_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = m_2L_2^2\dot{\phi}_2 + m_2L_1L_2\dot{\phi}_1$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2}\right) = m_2L_2^2\ddot{\phi}_2 + m_2L_1L_2\ddot{\phi}_1$$

$$(m_1+m_2)L_1^2\ddot{\phi}_1 \; + \; m_2 L_1 L_2 \ddot{\phi}_2 \; + \; (m_1+m_2) g L_1 \phi_1 \; = \; 0$$

$$m_2 L_2^2 \ddot{\phi}_2 + m_2 L_1 L_2 \ddot{\phi}_1 \; + \; m_2 g L_2 \phi_2 \; = \; 0$$

Since this is Taylor ch11, we want to find normal modes! To do that, we first write these two equations in matrix form. **Try it!**

$$\underline{\underline{M}}\,\ddot{\underset{\sim}{\phi}} = -\underline{\underline{K}}\,\underset{\sim}{\phi}$$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Then we want to find "normal modes" that behave like simple oscillators:

$$\ddot{\underset{\sim}{\phi}} = -\omega^2 \underset{\sim}{\phi}$$

ϕ_1 EOM: $(m_1+m_2)L_1^2\ddot{\phi}_1 + m_2L_1L_2\ddot{\phi}_2 + (m_1+m_2)gL_1\phi_1 = 0$

ϕ_2 EOM: $m_2L_2^2\ddot{\phi}_2 + m_2L_1L_2\ddot{\phi}_1 + m_2gL_2\phi_2 = 0$

$$\begin{pmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = - \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\underline{\underline{M}} \quad \underset{\sim}{\ddot{\phi}} \quad = - \underline{\underline{K}} \; \underset{\sim}{\phi}$$

want to find "normal modes" that behave like simple oscillators:

$$\underset{\sim}{\ddot{\phi}} = -\omega^2 \underset{\sim}{\phi}$$

$$\underline{\underline{M}}\left(-\omega^2 \underset{\sim}{\phi}\right) = -\underline{\underline{K}}\,\underset{\sim}{\phi} \Rightarrow \left(\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right)\underset{\sim}{\phi} = 0$$

true for nonzero $\underset{\sim}{\phi}$ only if $\left|\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right| = 0$

To keep algebra under control, let $m_1 = m_2 = m$
$L_1 = L_2 = L$

To keep algebra under control, let $m_1 = m_2 = m$
$L_1 = L_2 = L$

$$\underline{\underline{M}} \to mL^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\underline{\underline{K}} \to mgL \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{\underline{K}} - \omega^2 \underline{\underline{M}} = mL \begin{pmatrix} 2g - 2L\omega^2 & -\omega^2 L \\ -\omega^2 L & g - \omega^2 L \end{pmatrix}$$

$$= mL^2 \begin{pmatrix} 2\left(\frac{g}{L} - \omega^2\right) & -\omega^2 \\ -\omega^2 & \left(\frac{g}{L} - \omega^2\right) \end{pmatrix}$$

To find eigenvalues, set determinant of above matrix equal to zero and solve for ω^2 (square of angular frequency). Get two solutions for two normal modes, corresponding to two degrees of freedom. **Try it!**

$$0 = \begin{vmatrix} 2\left(\frac{g}{L}-\omega^2\right) & -\omega^2 \\ -\omega^2 & \left(\frac{g}{L}-\omega^2\right) \end{vmatrix} = 2\left(\frac{g}{L}-\omega^2\right)^2 - \omega^4 = 0$$

$$\pm\,\omega^2 = \frac{g\sqrt{2}}{L} - \omega^2\sqrt{2}$$

1 $$(+) \rightarrow \omega^2\left(1+\sqrt{2}\right) = \sqrt{2}\,\frac{g}{L} \Rightarrow \omega^2 = \frac{\sqrt{2}}{\sqrt{2}+1}\,\frac{g}{L}$$

$$\Rightarrow \omega_1 \cong 0.7654\sqrt{\frac{g}{L}}$$

2 $$(-) \rightarrow \omega^2\left(\sqrt{2}-1\right) = \sqrt{2}\,\frac{g}{L} \Rightarrow \omega^2 = \frac{\sqrt{2}}{\sqrt{2}-1}\,\frac{g}{L}$$

$$\Rightarrow \omega_2 \simeq 1.848\sqrt{\frac{g}{L}}$$

Next, find the corresponding eigenvectors. **Try it!**

Eigenvectors:

(1) $$(\underline{\underline{K}} - \omega^2 \underline{\underline{M}})\begin{pmatrix}1\\ b\end{pmatrix} = \underset{\sim}{0}$$

$$\begin{pmatrix} 2\frac{g}{L}\left(1-\frac{\sqrt{2}}{\sqrt{2}+1}\right) & -\frac{g}{L}\frac{\sqrt{2}}{\sqrt{2}+1} \\ -\frac{g}{L}\frac{\sqrt{2}}{\sqrt{2}+1} & \frac{g}{L}\left(1-\frac{\sqrt{2}}{\sqrt{2}+1}\right)\end{pmatrix}\begin{pmatrix}1\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix} 2\left(\frac{\sqrt{2}+1-\sqrt{2}}{\sqrt{2}+1}\right) & -\frac{\sqrt{2}}{\sqrt{2}+1} \\ -\frac{\sqrt{2}}{\sqrt{2}+1} & \frac{\sqrt{2}+1-\sqrt{2}}{\sqrt{2}+1}\end{pmatrix}\begin{pmatrix}1\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$2 - b\sqrt{2} = 0 \Rightarrow b = \sqrt{2} \Rightarrow \underset{\sim}{A}_1 = \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$$

② $$\frac{g}{L}\begin{pmatrix} 2\left(1-\frac{\sqrt{2}}{\sqrt{2}-1}\right) & -\frac{\sqrt{2}}{\sqrt{2}-1} \\ -\frac{\sqrt{2}}{\sqrt{2}-1} & 1-\frac{\sqrt{2}}{\sqrt{2}-1} \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

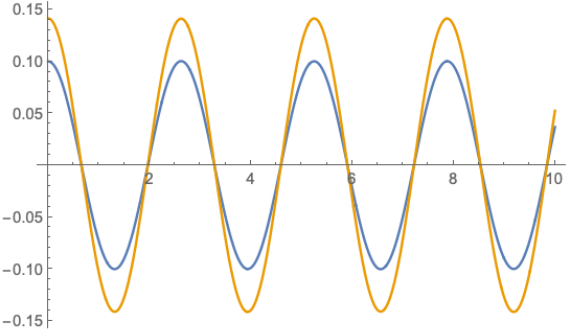
$$\begin{pmatrix} 2(\sqrt{2}-1-\sqrt{2}) & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2}-1-\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

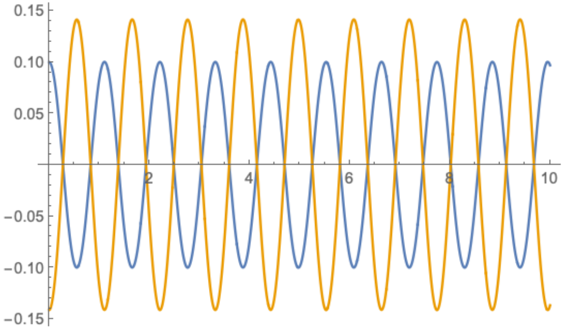
$$-2-\sqrt{2}\,b = 0 \Rightarrow b = -\sqrt{2} \Rightarrow \underset{\sim}{A}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

Illustrate with Mathematica!
(Do I make the period obvious on my $\phi(t)$ graphs?)

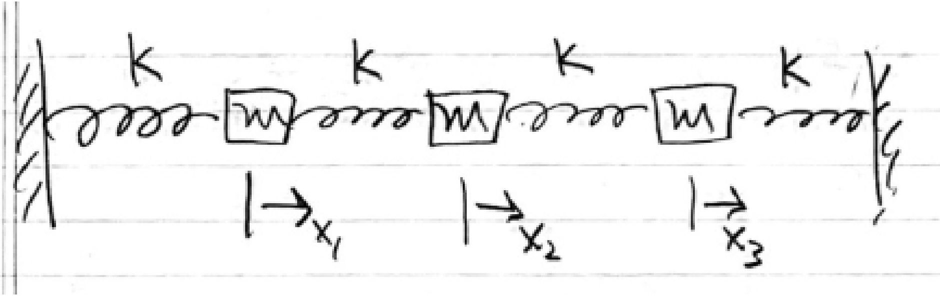
```
L1 = 1 ; L2 = 1 ; m1 = 1 ; m2 = 1; g = 9.8;
a = 0.1; b = +Sqrt[2]; tmax = 10;
soln = NDSolve[
    {eom1, eom2,
     ϕ1'[0] == 0, ϕ2'[0] == 0,
     ϕ1[0] == a, ϕ2[0] == a b},
    {ϕ1[t], ϕ2[t]},
    {t, 0, tmax}];
phi1[t_] := Evaluate[ϕ1[t] /. First[soln]]
phi2[t_] := Evaluate[ϕ2[t] /. First[soln]]
Plot[{phi1[t], phi2[t]}, {t, 0, tmax}]
```



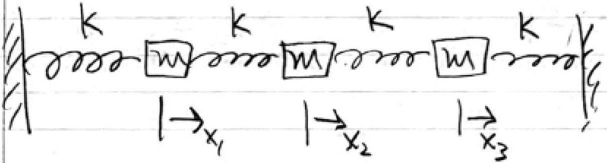
```
L1 = 1 ; L2 = 1 ; m1 = 1 ; m2 = 1; g = 9.8;
a = 0.1; b = -Sqrt[2]; tmax = 10;
soln = NDSolve[
    {eom1, eom2,
     ϕ1'[0] == 0, ϕ2'[0] == 0,
     ϕ1[0] == a, ϕ2[0] == a b},
    {ϕ1[t], ϕ2[t]},
    {t, 0, tmax}];
phi1[t_] := Evaluate[ϕ1[t] /. First[soln]]
phi2[t_] := Evaluate[ϕ2[t] /. First[soln]]
Plot[{phi1[t], phi2[t]}, {t, 0, tmax}]
```



Next example! Three coupled oscillators. Let all 3 masses equal m and let all four spring constants equal k.



Write T. Then write U. Then write $\mathcal{L} = T - U$. **Try it!**



$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)$$

$$U = \frac{1}{2}k\left[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2\right]$$

$$= \frac{1}{2}k\left[x_1^2 + x_2^2 - 2x_1x_2 + x_1^2 + x_3^2 - 2x_2x_3 + x_2^2 + x_3^2\right]$$

$$= \frac{1}{2}k\left[2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3\right]$$

$$U = k\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right)$$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - k\left(x_1^2 + x_2^2 + x_3^2\right) + k\left(x_1x_2 + x_2x_3\right)$$

Now find the three EOMs and write them in matrix form. **Try it!**

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) - k\left(x_1^2 + x_2^2 + x_3^2\right) + k\left(x_1 x_2 + x_2 x_3\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m\ddot{x}_1 \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m\ddot{x}_2 \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_3} = m\ddot{x}_3$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \boxed{-2kx_1 + kx_2 = m\ddot{x}_1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \boxed{-2kx_2 + k(x_1 + x_3) = m\ddot{x}_2}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = \boxed{-2kx_3 + kx_2 = m\ddot{x}_3}$$

$$m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\ddot{\underset{\sim}{x}} = -\frac{k}{m}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\underset{\sim}{x}$$

$$\ddot{\underset{\sim}{x}} = -\omega^2 \underset{\sim}{x} \Rightarrow \omega^2 \underset{\sim}{x} = \frac{k}{m}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\underset{\sim}{x}$$

let $\omega^2 = \frac{k}{m}\lambda \quad \Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$

Now find the three eigenvalues. It's easier to work with the dimensionless matrix above. You will find 3 dimensionless eigenvalues λ_1, λ_2 λ_3. Then the natural frequency ω_i of the corresponding normal mode is given by $\omega_i^2 = (k/m)\,\lambda_i$. **Try it!**

let $\omega^2 = \frac{K}{m}\lambda \quad \rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^3 + 0 + 0 - (2-\lambda) - (2-\lambda) - 0 = 0$

$(2-\lambda)^3 = 2(2-\lambda)$

$(\lambda-2)^3 = 2(\lambda-2)$

$\lambda = 2 \quad \text{or} \quad (\lambda-2)^2 = 2$

$\lambda = 2 \quad \text{or} \quad \lambda - 2 = \pm\sqrt{2}$

$\lambda = 2 \quad \text{or} \quad \lambda = 2 \pm \sqrt{2}$

$\lambda_1 = 2 - \sqrt{2} \qquad \lambda_2 = 2 \qquad \lambda_3 = 2 + \sqrt{2}$

$\omega_1^2 = \frac{K}{m}(2-\sqrt{2}) \qquad \omega_2^2 = 2\frac{K}{m} \qquad \omega_3^2 = (2+\sqrt{2})\frac{K}{m}$

$\omega_1 \simeq 0.7654\sqrt{\frac{K}{m}} \qquad \omega_2 \simeq 1.414\sqrt{\frac{K}{m}} \qquad \omega_3 \simeq 1.848\sqrt{\frac{K}{m}}$

Next, find the eigenvector for λ_1. **Try it!**

Meanwhile I'll find it on the demo setup. (About 0.416 Hz.)

(1)

$$\begin{pmatrix} 2-(2-\sqrt{2}) & -1 & 0 \\ -1 & 2-(2-\sqrt{2}) & -1 \\ 0 & -1 & 2-(2-\sqrt{2}) \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sqrt{2} - a = 0 \Rightarrow a = \sqrt{2}$$

$$-a + \sqrt{2}\, b = 0 \Rightarrow \sqrt{2}(-1+b) = 0 \Rightarrow b = 1$$

$$\underset{\sim}{A}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Next, find the eigenvector for λ_2. **Try it!**
Meanwhile I'll find it on the demo setup. (About 0.768 Hz.)

(2)

$$\begin{pmatrix} 2-2 & -1 & 0 \\ -1 & 2-2 & -1 \\ 0 & -1 & 2-2 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$a = 0 \qquad -1 - b = 0 \Rightarrow b = -1$$

$$\underset{\sim}{A}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Next, find the eigenvector for λ_3. **Try it!**
Meanwhile I'll find it on the demo setup. (About 0.98 Hz.)

③ $\begin{pmatrix} 2-(2+\sqrt{2}) & -1 & 0 \\ -1 & 2-(2+\sqrt{2}) & -1 \\ 0 & -1 & 2-(2+\sqrt{2}) \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\sqrt{2} - a = 0 \Rightarrow a = -\sqrt{2}$$

$$-a - \sqrt{2}b = 0 \Rightarrow +\sqrt{2} - \sqrt{2}b = 0 \Rightarrow 1 - b = 0$$

$$\Rightarrow b = 1$$

$$\lambda_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

DEMO!

lowest mode

frequency $f_1 \simeq 0.416\,\mathrm{Hz}$

---

next mode: frequency $f_2 = 1.85 f_1$

$\simeq 0.768\,\mathrm{Hz}$

This mode is easiest to find via free oscillations, as initial state is easy to arrange.

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highest mode: frequency $f_3 = 2.41 f_1$

$\simeq 0.98\,\mathrm{Hz}$

I made some videos, once we found the three normal-mode frequencies for the demonstration setup. (We should make longer videos, with a solid backdrop, before we disassemble the demo.)

http://positron.hep.upenn.edu/p351/files/coupled_3_osc_mode_1_sloshing.mp4

http://positron.hep.upenn.edu/p351/files/coupled_3_osc_mode_2_breathing.mp4

http://positron.hep.upenn.edu/p351/files/coupled_3_osc_mode_3_beating.mp4

Physics 3351 — Wednesday, April 5, 2023

- Remember to **READ** §11.5 ("the general case") on your own, as it is worth seeing, but I didn't think our going through it together on the board would add much value.
- These slides will be on Canvas/files/notes as well as at

`http://positron.hep.upenn.edu/p351/files/p351_notes_20230405.pdf`

- Mathematica notebook (and exported PDF):

  `http://positron.hep.upenn.edu/p351/files/phys3351_20230405.nb`

  `http://positron.hep.upenn.edu/p351/files/phys3351_20230405.pdf`

- I am in DRL 1W15 or at `ashmansk@hep.upenn.edu` .