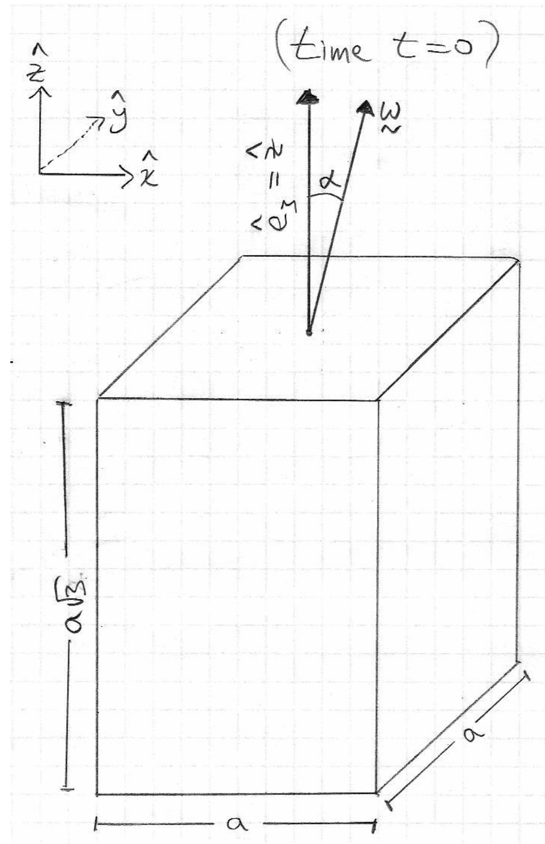
Problem 1.

A uniform rectangular solid of mass m and dimensions $a \times a \times a\sqrt{3}$ (volume $\sqrt{3}\ a^3$) is allowed to undergo torque-free rotation. At time $t = 0$, the long axis (length $a\sqrt{3}$) of the solid is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\hat{\boldsymbol{z}} + \sin\alpha\hat{\boldsymbol{x}})$.

(a) Show (or argue) that the inertia tensor has the form $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and find the constant I_0.



(b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. Write $\boldsymbol{L}(t = 0)$ both in terms of $\hat{\boldsymbol{e}}_1$,$\hat{\boldsymbol{e}}_2$,$\hat{\boldsymbol{e}}_3$ and in terms of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. Which of these two expressions will continue to be valid into the future?

(c) Draw a sketch showing the vectors $\hat{\boldsymbol{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for egg-shaped ("prolate") objects ($\lambda_3 < \lambda_1$) than it is for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$).

(d) Draw and label the "body cone" and the "space cone" on your sketch.

(e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. Be careful with the "sign" of the $\boldsymbol{\Omega}_{\mathbf{body}}$ vector, i.e. be careful not to draw $-\boldsymbol{\Omega}_{\mathbf{body}}$ when you mean to draw $\boldsymbol{\Omega}_{\mathbf{body}}$.

(f) You argued in HW11 that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$.

(g) In the $\alpha \ll 1$ limit (so $\tan\alpha \approx \alpha$, $\tan(2\alpha) \approx 2\alpha$, etc.), find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{e}}_3$ during subsequent motion of the solid. (This should be some constant factor times α.) A simple argument is sufficient here, no calculation.

(h) At what time t is this maximum deviation first reached?

(This problem shows that for an American-football-like object, the frequency of the wobbling motion is smaller than the frequency of the spinning motion — which is opposite the conclusion that you reached for the flying dinner plate, whose wobbling was twice as fast as its spinning.)

Possibly useful equations.

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\Omega} + m\,(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} = \boldsymbol{F} + 2m\,\boldsymbol{v} \times \boldsymbol{\Omega} + m\Omega^2 \rho\,\hat{\boldsymbol{\rho}}$$

For a uniform solid cylinder of radius R about its symmetry axis, $I = mR^2/2$. For a uniform thin rod of length L about its center (perpendicular to the rod axis), $I = mL^2/12$. For a rectangular plate about its center (rotation axis normal to plate), $I = m(a^2 + b^2)/12$, where a and b are the short and long side lengths.

For a free symmetric top, $\boldsymbol{\Omega}_s = \boldsymbol{L}/\lambda_1$

Euler equations:

$$\tau_1 = \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3$$
$$\tau_2 = \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1$$
$$\tau_3 = \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2$$

If $\boldsymbol{\tau} = 0$ and $\lambda_1 = \lambda_2$ then the Euler equations reduce to the simpler form

$$\dot{\omega}_3 = \frac{\lambda_1 - \lambda_1}{\lambda_3}\,\omega_1\omega_2 = 0$$

$$\dot{\omega}_1 = \frac{\lambda_1 - \lambda_3}{\lambda_1}\,\omega_2\omega_3 = -\left(\frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3\right)\omega_2 = -\,\Omega_b\,\omega_2$$

$$\dot{\omega}_2 = \frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3\omega_1 = \left(\frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3\right)\omega_1 = \Omega_b\,\omega_1$$

so we can represent the precession of the $\boldsymbol{\omega}$ vector as an angular velocity vector $\boldsymbol{\Omega}_b$ with

$$\boldsymbol{\Omega}_b = \left(\frac{\lambda_3 - \lambda_1}{\lambda_1}\,\omega_3\right)\hat{\boldsymbol{e}}_3.$$

Euler-angle convention: Start with body axes aligned with space axes. (i) Rotate body through angle ϕ about $\hat{\boldsymbol{z}}$. This leaves $\hat{\boldsymbol{e}}_3$ alone but rotates the first and second body axes in the xy plane. In particular, the second body axis now points in a direction called $\hat{\boldsymbol{e}}_2'$. (ii) Rotate body through angle θ about the new axis $\hat{\boldsymbol{e}}_2'$. This moves the body axis $\hat{\boldsymbol{e}}_3$ to the direction whose polar angles are θ and ϕ. (iii) Rotate the body about $\hat{\boldsymbol{e}}_3$ through whatever angle ψ is needed to bring the body axes $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_1$ into their assigned directions.

At any instant, you can use the values of ϕ,θ,ψ at that instant to write each body unit vector $\hat{\boldsymbol{e}}_i$ as a linear combination of $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{y}}$,$\hat{\boldsymbol{z}}$. The coefficients involve sines and cosines of ϕ,θ,ψ but have no explicit time dependence.